Poster at XI Brazilian School of Probability.

Maresias, SP, August 5-11, 2007

Substitution operators:

All or them are quasi-local, therefore continuous, therefore all of them have an invariant measure. In addition, all of them are fine, which helps to localize their invariant measure and to prove its non-uniqueness.

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Random processes with discrete time are usually defined by some *transition operator* or just operator, which transforms the measure at any time step into the measure at the next time step. A measure μ is called *invariant* for an operator P if $\mu = \mu P$. (We write operators between measures and events, therefore on the right side of measures.) Existence and uniqueness of an invariant measure are among the most important features of an operator. From the mathematical point of view, the problem of existence of invariant measure is a special case of the well-known fixed-point problem.

The most usual way to prove existence of invariant measure of a random

process is to take an arbitrary initial measure, iteratively apply to it the transition operator P , prove that the Cesàro transformation of the resulting sequence has at least one accumulation point and prove that this point is invariant for P . However, all versions of this method work only for linear operators.

Here we prove existence of invariant measure for a larger class of operators of random processes without assuming their linearity. This allows us to apply this proof to some variable-length processes (like those studied in [T.2004]), whose operators are mostly non-linear.

Let us take any set Ω and a countable algebra $\mathbf{A}$ of its subsets. We denote by $\sigma(\mathbf{A})$ the minimal σ -algebra, which contains $\mathbf{A}$. Let $\mathcal{M}$ be the set of normalized measures (or probability distributions, which is the same) on $\sigma(\mathbf{A})$. Let us say that a sequence of measures $\mu_n \in \mathcal{M}$ *tends* or *converges* to a measure $\lambda \in \mathcal{M}$ if $\mu_n(S)$ tends to $\lambda(S)$ for every $S \in \mathbf{A}$.

In fact, we shall deal with some $\mathcal{M}' \subset \mathcal{M}$ and call maps from $\mathcal{M}'$ to $\mathcal{M}'$ *operators*. We say that an operator $P : \mathcal{M}' \rightarrow \mathcal{M}'$ is *continuous* if whenever a sequence $\mu_n \in \mathcal{M}'$ tends to $\lambda \in \mathcal{M}'$, the sequence $\mu_n P$ tends to λP . (The well-known sequential continuity.) We call a set $\mathcal{M}' \subset \mathcal{M}$ *compact* if every sequence $\mu_n \in \mathcal{M}'$ has a subsequence, which converges to an element of $\mathcal{M}'$. We say that a set $\mathcal{M}' \subset \mathcal{M}$ is *convex* if for any $\mu, \nu \in \mathcal{M}'$ and any $k \in [0, 1]$ the measure $k \cdot \mu + (1 - k) \cdot \nu$ also

belongs to $\mathcal{M}'$. A measure $\mu \in \mathcal{M}'$ is called *invariant* for an operator P if $\mu = \mu P$.

Theorem 1. *For any non-empty compact convex $\mathcal{M}' \subset \mathcal{M}$, any continuous operator $P : \mathcal{M}' \rightarrow \mathcal{M}'$ has an invariant measure.*

The proof is based on Schauder-Tychonoff theorem:

$$\left. \begin{array}{l} \text{Let } \mathcal{L} \text{ be a separated locally convex topological linear} \\ \text{space, } \mathcal{K} \text{ a non-void compact convex subset of } \mathcal{L}, \\ P \text{ any continuous map of } \mathcal{K} \text{ into itself. Then } P \\ \text{admits at least one fixed point.} \end{array} \right\} \quad (1)$$

See e.g. [Edwards, section 3.6].

Here continuity and compactness are in the topology on that space restricted to that set.

For any $\mathcal{M}' \subset \mathcal{M}$, any natural number n and any sets $S_1, \dots, S_n \in \mathbf{A}$, let us define the set

$$\text{Dom}(S_1, \dots, S_n | \mathcal{M}') \subset [0, 1]^n$$

as follows:

$$\text{Dom}(S_1, \dots, S_n | \mathcal{M}') \stackrel{\text{def}}{=} \{(\mu(S_1), \dots, \mu(S_n)) : \mu \in \mathcal{M}'\}.$$

In other words, an n -tuple

$$(x_1, \dots, x_n) \in [0, 1]^n$$

belongs to $\text{Dom}(S_1, \dots, S_n | \mathcal{M}')$ if and only if there is $\mu \in \mathcal{M}'$ such that

$$\mu(S_1) = x_1, \dots, \mu(S_n) = x_n.$$

Every set $\text{Dom}(S_1, \dots, S_n | \mathcal{M}')$ is compact.

For any $\mathcal{M}' \subset \mathcal{M}$ we call an operator $P : \mathcal{M}' \rightarrow \mathcal{M}'$ *quasi-local* if for any $S \in \mathbf{A}$ and any $\varepsilon > 0$ there is a natural number n , sets $S_1, \dots, S_n \in \mathbf{A}$ and a continuous function

$$f : \text{Dom}(S_1, \dots, S_n | \mathcal{M}') \rightarrow [0, 1]$$

such that

$$\forall \mu \in \mathcal{M}' : |\mu P(S) - f(\mu(S_1), \dots, \mu(S_n))| < \varepsilon. \quad (2)$$

Theorem 2. *For any compact $\mathcal{M}' \subset \mathcal{M}$, any quasi-local operator $P : \mathcal{M}' \rightarrow \mathcal{M}'$ is continuous.*

Now let us go to a still narrower class of operators. For any $\mathcal{M}' \subset \mathcal{M}$, we call an operator $P : \mathcal{M}' \rightarrow \mathcal{M}'$ *local* if for any $S \in \mathbf{A}$ there is a natural number n , sets $S_1, \dots, S_n \in \mathbf{A}$ and a continuous function

$$f : \text{Dom}(S_1, \dots, S_n | \mathcal{M}') \rightarrow [0, 1]$$

such that

$$\forall \mu \in \mathcal{M}' : \mu P(S) = f(\mu(S_1), \dots, \mu(S_n)). \quad (3)$$

Starting here, we consider only the special case, in which Ω is a bi-infinite product $A^{\mathbb{Z}}$, where A is a non-empty finite set, which we call *alphabet*.

Elements of the alphabet are called *letters*. A generic element of Ω is a bi-infinite sequence $x = (\dots, x_{-1}, x_0, x_1, \dots)$, where all $x_i \in A$. Taking the discrete topology on A , we obtain the algebra $\mathbf{A}$ as the minimal algebra, which contains all the sets of the form

$$\{x \in \Omega \mid x_{i+1} = a_1, \dots, x_{i+n} = a_n\}, \quad (4)$$

where $i \in \mathbb{Z}$ and $a_1, \dots, a_n \in A$. The class of sets (4) also serves as a base for the product topology on Ω , in which Ω is compact due to Tychonoff compactness theorem. Hence the set $\mathcal{M}$ of normalized measures on $\sigma(\mathbf{A})$ is compact also.

We call a normalized measure on Ω *shift-invariant* if it is invariant under all shifts along $\mathbb{Z}$. We shall use the abbreviation “s.i.n. measures” for shift-invariant normalized measures. We denote by $\mathcal{M}_A$ the set of s.i.n. measures on $A^{\mathbb{Z}}$. Notice that $\mathcal{M}_A$ is closed in $\mathcal{M}$, whence it is compact.

Any finite sequence of letters is called a *word*. The *length* of a word W , denoted by $|W|$, is the number of letters in it. Any letter may be considered as a word of length one. There is the empty word, denoted by Λ , whose length is zero. We assume that comma and brackets never belong to our alphabet and if we write several words and letters one after another, perhaps separated by commas or included in brackets, they form one word (commas and brackets eliminated), which we call their *concatenation*. Dealing with s.i.n. measures, we may use the following

simplified notation for any word $W = (a_1, \dots, a_n)$:

$$\mu(W) = \mu(a_1, \dots, a_n) = \mu(x_{i+1} = a_1, \dots, x_{i+n} = a_n). \quad (5)$$

Due to shift-invariance of μ , the probability (5) does not depend on i and we call it the *frequency* of the word W in the measure μ . Any s.i.n. measure on Ω is determined by its values (5) on all words in the alphabet A . To form a s.i.n. measure, all the numbers $\mu(W)$ must be non-negative, μ on the empty word must equal one and for any letter a and any word W (including the empty one) must be

$$\mu(W) = \sum_{a \in A} \mu(W, a) = \sum_{a \in A} \mu(a, W),$$

where (W, a) and (a, W) are concatenations of the word W and letter a in the two possible orders. Given two words W, V , where $|W| \leq |V|$, we say that W *enters* V *at a position* k if $1 \leq k \leq |V| - |W| + 1$ and W coincides with the word consisting of those letters of V , which occupy positions from k to $k + |W| - 1$ in it. We call a word W *self-overlapping* if there is a word V such that $|V| < 2 \cdot |W|$ and W enters V at two different positions. A word is called *self-avoiding* if it is not self-overlapping.

Now we are ready to speak about a class of non-linear operators, which we call *substitution operators*. They are similar to *variable length operators*, which we discussed in [T.2002, T.2004].

A generic substitution operator is denoted by $(G \xrightarrow{\rho} H)$, where G and H are self-avoiding words and ρ is a real number in $[0, 1]$. Informally

speaking, this operator substitutes every entrance of the word G in every configuration by the word H with a probability ρ or leaves it unchanged with a probability $1 - \rho$ independently of states and fate of the other components. However, we define only how our operators act on s.i.n. measures, but not on configurations.

If we studied processes with continuous time, we could deal with all words rather than self-avoiding ones. However, even so there are not so few of them. If the alphabet A has $|A|$ letters, there are $|A|^n$ words of length n and more than

$$\frac{|A| - 2}{|A| - 1} \cdot |A|^n$$

of them are self-avoiding. Thus the fraction of self-avoiding words among n -letter words for any n tends to ∞ when $|A| \rightarrow \infty$. For the small values of n , the empty word is self-avoiding, all one-letter words are self-avoiding and there are $|A|(|A| - 1)$ self-avoiding words among two-letter words.

We have to avoid the *bad case*, in which

$$\text{Bad case: } G \text{ is non-empty, } H \text{ is empty and } \rho = 1. \quad (6)$$

In the bad case the substitution operator cannot be applied to the measure concentrated in the bi-infinite concatenations $\dots, G, G, G, \dots$ of the word G . In all the other cases a substitution operator can be applied to all s.i.n. measures. For every substitution operator and every $\mu \in \mathcal{M}_A$, we define a *coefficient of extension* or just *extension* for short, denoted

by Ext , which equals

$$\text{Ext} = 1 + \rho \cdot (|H| - |G|) \cdot \mu(G). \quad (7)$$

Informally speaking, extension is that coefficient, by which is multiplied the length of a long word distributed according to the given measure μ in result of action of the operator in question.

Now we are going to define several specific classes of substitution operators. In every case P denotes the operator in question, $\mu \in \mathcal{M}_A$ denotes an arbitrary s.i.n. measure and μP denotes the result of application of P to μ .

Conversion: $(g \xrightarrow{\rho} h)$ is the only linear operator in our list. Given two different letters $g, h \in A$, *conversion of g into h* is a map from $\mathcal{M}_A$ to $\mathcal{M}_A$. Informally, conversion means that every occurrence of the letter g is either substituted by h with a probability $\rho \in [0, 1]$ or left unchanged with a probability $1 - \rho$ independently from presence and fate of other occurrences. The extension in this case equals one.

Compression: $(G \xrightarrow{1} h)$. Given a non-empty self-avoiding word G in an alphabet A and a letter $h \notin A$, *compression of G into h* is the following map from $\mathcal{M}_A$ to $\mathcal{M}_{A'}$, where $A' = A \cup \{h\}$: every occurrence of the word G is substituted by the letter h with probability 1.

The extension in this case equals

$$\text{Ext} = 1 - (|G| - 1) \cdot \mu(G) \geq \frac{1}{|G|}.$$

Decompression: $(g \xrightarrow{1} H)$. Informally speaking, given a non-empty word H in the alphabet A and a letter $g \notin A$, *decompression of g into H* is the following map from $\mathcal{M}_{A'}$ to $\mathcal{M}_A$, where $A' = A \cup \{g\}$: every occurrence of the letter g turns into the word H with probability 1. The extension in this case equals

$$\text{Ext} = 1 + (|H| - 1) \cdot \mu(g).$$

Insertion: $(\Lambda \xrightarrow{\rho} h)$. Informally, insertion of a letter $h \notin A$ into a measure in the alphabet A with a rate $\rho \in [0, 1]$ means that a letter h is inserted with the probability ρ between every two neighbor letters independently from other places. The extension in this case equals $1 + \rho$.

Deletion: $(g \xrightarrow{\rho} \Lambda)$. Informally, deletion of a letter $g \in A$ from a measure in the alphabet A with a rate $\rho \in [0, 1)$ means that every occurrence of g either disappears with a probability ρ or remains unchanged with a probability $1 - \rho$ independently from other occurrences. The extension in this case equals $1 - \rho \cdot \mu(g)$.

Now let us speak about continuity of these operators. Compression, Decompression, Conversion and Insertion are evidently local, therefore

continuous for all $\rho \in [0, 1]$. Deletion is not local, but we can prove that it is quasi-local for all $\rho < 1$.

We conclude that all the operators defined above are continuous, whence all their finite superpositions are continuous also. Let us show that these superpositions include a large variety of possibilities. For example, if we want some non-empty self-avoiding word H to appear with a rate ρ into a measure in an alphabet A , we may first use Insertion to make some special letter $g \notin A$ appear with the rate ρ and then use Decompression to turn every occurrence of g into H . If we want some non-empty self-avoiding word G to disappear with a rate $\rho < 1$ from a measure in an alphabet A , we may first use Compression to turn every occurrence of G into some special letter $h \notin A$, then use Deletion to make every occurrence of h disappear with the rate ρ and finally use Decompression to expand the remaining occurrences of h back into G . Finally, if we have two non-empty self-avoiding words G and H in an alphabet A and want G to turn into H with a rate $\rho < 1$, we may first use Compression to turn every occurrence of G into some special letter $g \notin A$, then use Conversion to turn every occurrence of g into another letter $h \notin A$ with the rate ρ and finally use Decompression two times to turn all the occurrences of g back into G and all the occurrences of h into H . Every one of these superpositions is continuous and therefore has an invariant measure due to theorem 1.

In [T.2004] we considered alphabet $A = \{\oplus, \ominus\}$, whose elements were called *plus* and *minus*, and two specific operators: *flip* denoted by Flip_β and *annihilation* denoted by Ann_α . Flip is a special case of Conversion. In our present notations, Flip_β is $(\ominus \xrightarrow{\beta} \oplus)$ as it turns every minus into plus with a probability β independently from the fate of other components. Annihilation Ann_α is $((\oplus, \ominus) \xrightarrow{\alpha} \Lambda)$ as it makes every entrance of the self-avoiding word $(\oplus, \ominus)$ disappear with a probability $\alpha < 1$ independently from the other components. We iteratively applied the superposition of these two operators (first flip, then annihilation) to the initial measure $\delta_\ominus$, concentrated in the configuration “all minuses” and denoted by μ_t the resulting sequence of measures:

$$\mu_t = \delta_\ominus (\text{Flip}_\beta \text{Ann}_\alpha)^t.$$

The main result of [T.2004] with a correction in [Ramos] was this:

$$\left. \begin{array}{l} \text{for all natural } t \text{ the frequency of pluses in} \\ \text{the measure } \mu_t \text{ does not exceed } 250 \cdot \beta / \alpha^2. \end{array} \right\} \quad (8)$$

Now we can prove more:

Theorem 3. *For all $\beta \in [0, 1]$ and $\alpha \in (0, 1)$ the operator $\text{Flip}_\beta \text{Ann}_\alpha$ has an invariant measure, whose frequency of pluses does not exceed $250 \cdot \beta / \alpha^2$.*

The proof of theorem 3 is based on the following property of substitution operators. For any two s.i.n. measures $\mu, \nu \in \mathcal{M}$ we denote by $[\mu, \nu]$

their convex hull:

$$[\mu, \nu] \stackrel{\text{def}}{=} \{k \cdot \mu + (1 - k) \cdot \nu \mid 0 \leq k \leq 1\}.$$

We call an operator $P : \mathcal{M}' \rightarrow \mathcal{M}'$ *fine* if

$$\lambda \in [\mu, \nu] \implies \lambda P \in [\mu P, \nu P].$$

Theorem 3 is the main result reported here. It is based on the previous theorems and the fact that all flips and annihilations are fine. In the nearest future we [Rocha] shall prove that all substitution operators are fine.

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