Scaling transformation of random walk and generalized statistics

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Abstract

We use a decimation procedure in order to obtain the dynamical renormalization group transformation (RGT) properties of random walk distribution in a $1+1$ lattice. We obtain an equation similar to the Chapman–Kolmogorov equation. First we show the existence of invariants of the RGT, and that the Tsallis distribution 

$$R_q(x) = [1 + b(q-1)x^2]^{(1-q)/(q-1)} (q > 1)$$

is a quasi-invariant of the RGT. We obtain the map $q' = f(q)$ from the RGT and show that this map has two fixed points: $q = 1$, attractor, and $q = 2$, repellor, which are the Gaussian and the Lorentzian, respectively. Finally we use those concepts to show that the nonadditivity of the Tsallis entropy needs to be reviewed. 

\section{1. Introduction}

Random walks have been one of the simplest approaches to statistical mechanics. On the other hand, scaling methods are essential in circumstances where a system is scaling invariant or acts as if it was so. As a general rule, whenever a characteristic control length diverges, a proper treatment requires renormalization group (RG) methods \cite{1–4}. For this purpose, scaling on linear lattices is a powerful and elegant tool, since these systems have recursive hierarchical geometry. In a previous article \cite{5} we analyzed the
presence of a random force in a lattice of lattice spacing $a$ using RG methods. There we got that the evolution of a probability distribution $P_n(x)$, according a RGT, is given by

$$P_{n+1}(x) = \int P_n(y)P_n(x - y)dy, \quad n \geq 1,$$

where $n$ is the order of iteration. The convolution equation (1) is a discrete form of the Chapman–Kolmogorov equation. Eq. (1) is iterative, and the function of order $n$, $P_n(x)$, for low $n$, may be very different from that of order $n - 1$. In this aspect, the left hand side of Eq. (1) may have random walk distributions (RWD) which are formally different from the right hand side.

The main objective of this article is to use that tool to analyze both the generalized forms of random walk distributions and the Tsallis rule for the nonadditivity of the entropy [6–9]. We arrive at two main conclusions. The first one, we confirm the powerful utility of the Tsallis distribution for systems which are not described by Gaussians. Second, for systems with long-range order Tsallis rule for the addition of entropies is an approximation and needs to be reviewed.

This article follows as: in Section 2 we discuss the invariants and the quasi-invariants of the RGT. As a consequence we show that Tsallis distributions are transformed according to a specific law, i.e., $q' = f(q)$. This implies a $q$ dependence on the system size. In Section 3 we call attention to a growth model where we fit the height of the deposited particles using a Tsallis curve. As a consequence, $q$ appears explicitly as a function of the number $N$ of particles. The Sections 2 and 3 lead us to the conclusion that $q$ is size dependent and $|q - 1|$ grows as $N \to \infty$. Finally, in Section 4, we use those concepts to discuss the nonadditivity of the Tsallis entropy.

2. Invariants and quasi-invariants of the RGT

In order to understand the evolution of the random walk as we change the scale, we shall denote invariants of the renormalization group transformation (IRGT) the functions which keep their form under two consecutive transformations [5] given by Eq. (1). Formally, the IRGT may be expressed as

$$P_n(x) = bP_{n-2}(bx),$$

in such a way that the lattice spacing $a$ transforms as $a' = 2a$. We shall see that the noise line width transform as $c' = c/b$.

By using direct integration we show that the Gaussian $\exp(-(x/c)^2)$, the Lorentzian $(c^2 + x^2)^{-1}$ and the delta function $\delta(x/c)$ are IRGT. Moreover, the set of Levy functions [10]

$$L(\mu, x) = \frac{1}{2\pi} \int e^{ikx}e^{-(ck)^\mu} dk$$

are IRGT. This can be directly obtained from Eqs. (1) and (2) with

$$b = 2^{-2/\mu}.$$
The Gaussian ($\mu = 2$), the Lorentzian ($\mu = 1$), and the delta function ($\mu = 0$) are particular cases of Eq. (3). The parameter $c$ scales in the same form as $a$ only for the Gaussian. That is, Gaussians are commensurate with the lattice. For fractional $0 < \mu < 2$, $L(\mu, x)$ is used in the study of fractal diffusion [11]. Recently Chaves [12] proposed a fractal diffusion equation to describe Lévy distributions. As a consequence he predicted the occurrence of asymmetric diffusion, which was confirmed experimentally [13] with $\mu \approx 1.3$.

Consider now the sequence of functions $P_n(q, x)$, where $q$ is a continuous parameter $1 \leq q \leq 2$, and $x$ is unbound: $|x| < \infty$. We introduce the sequence with the function

$$P_1(q, x) = R_q(x),$$

(5)

where

$$R_q(x) = A_q[1 + \beta(q - 1)x^2]^{1/q}.$$  

(6)

For large $x$ and $q \neq 1$ Eq. (6) behaves as a power law. It decays more slowly than Gaussians ($q = 1$) and is more appropriate to the study of critical behavior [6–9]. This RWD has been applied successfully to the study of several phenomena such as turbulence [14], anomalous random walk [15], linear response [16] and to scaling properties of multifractal attractors [17]. This distribution is as well a solution of a non-linear Fokker Planck equation [18].

Using Eq. (6) in Eq. (1) we obtain a new function $P_2(q, x)$, which in general does not have the same form $R_q(x)$. This is not a surprise since the Tsallis recipe is for non-Markovian systems which do not satisfy the Chapman–Kolmogorov equation. However, $P_1(q, x)$ is very close to $R_q(x)$ if the parameters are adjusted to become new parameters $q', A'_q$ and $\beta'$. Thus we can say

$$P_2(q, x) \approx P_1(q', x) = R_{q'}(x),$$

(7)

or successively $P_n(q, x) = P_{n-1}(q', x)$. We shall call this RGT quasi-invariant (QRGT). We suggest a procedure to obtain the transformation $q' = f(q)$ by making the functions and all derivatives up to the fifth order to agree at the origin. From those we obtain

$$q' = f(q) = q - \frac{4q(q - 1)(q - 2)}{3q^2 - 16q + 5}.$$  

(8)

We shall notice that this result is exact only for $q = 1$, Gaussian, and $q = 2$, Lorentzian. Consequently, $f$ is an approximate function. However, this is a good approximation [5].

In Fig. 1 we plot $q'$ as a function of $q$. The straight line, curve 1(a), is the set of the fixed points $q' = q$, while curve 1(b) is the map $q' = f(q)$. The map shows two fixed points $q^* = f(q^*)$ at $q_1^* = 1$ and $q_2^* = 2$. We shall notice that the requirement for stability $|df(q^*)/dq| < 1$ is fulfilled only for $q^* = 1$. Consequently, only the Gaussian is a stable fixed point (attractor). Curve 1(c) shows the trajectory obtained from the return map.

Mendes and Tsallis [19], using a RG approach for a gas Hamiltonian, arrived at similar conclusion, i.e., that $q^* = 1$, is a fixed stable point and they postulated the existence of an unstable fixed point $q^* > 1$. For realistic systems with strong correlation,
Fig. 1. The return map. We plot $q'$ as a function of $q$ in the interactive process. In (a) we show the curve $q' = q$. In (b) we show the curve $q' = f(q)$ with the fixed points $q^* = f(q^*)$ at $q_1^* = 1$ and $q_2^* = 2$. In (c) we show a trajectory going from an arbitrary $q \neq q^*$, approaching successively the stable fixed point.

it is possible that other fixed points arise. However, that is a speculation without a proof. In the next section we see a possible fixed point $q^*$, which is neither 1 nor 2.

3. Scaling and growth

In order to analyze the scaling result in a more specific problem we focus our attention on the growth model proposed by Cordeiro et al. [20]. There a deposition process is described where a punctual source drops unity particles at the origin $x = 0$, from a height $H$. That is, the particles start the motion at the coordinate $(0, H)$ of the $xy$ plane. At each step down the particle moves randomly in the horizontal. At the beginning, the particle will give $H - 1$ steps down, and $H - 1$ steps in the horizontal with equal probability to the left or to the right. As the pile grows the falling particles will have their phase space decreased since some steps are not allowed [20]. Some conclusions obtained from this model can be cast in simple power laws. For example, the “mean square displacement”

$$\langle x^2 \rangle = aH,$$

and the number of the deposited particles

$$N = bH^{3/2},$$

are well defined power laws of the height $H$. To explain these results, and to fit the data for the density of deposited particles $n(x)$ we tried both Gaussians and the Tsallis distribution given by

$$n(x) \propto (R,q(x))^q,$$
where \( R_q \) is given by Eq. (6). The Tsallis distribution fits better all the computer experiments than the Gaussians. It is a surprise that an anomalous RW, Eq. (6), yields the same power law, Eqs. (9) and (10), as the normal RW. Moreover, the coefficients \( a \) and \( b \) are more precise when we use Eq. (11). This shows that Eq. (6) is a powerful fitting ansatz.

There are two non-intuitive results in the above analysis: first, the parameter \( q > 1 \) is an increasing function of the number of deposited particles, i.e., \( q = q(N) \), and for \( N_1 > N_2 \), then \( q_1 > q_2 \); second, as a consequence of the first one, there is a violation of the law of great numbers, since as \( N \) increases the system gets away from the Gaussian. It seems as well that as \( N \to \infty \), \( q \to q^* \) with \( q^* \approx 1.82 \).

The second possibility sounds bizarre, however it is the main consequence of describing systems with long range correlation with distributions such as Eq. (6). Since, according to Tsallis, \( q \) is a measure of this long range correlation, when we add more particles the distribution becomes less and less Gaussian. Consequently, as \( N \) increases \( q \) increases. Before we continue we shall ask ourselves a central question: is the process described by Cordeiro et al. \cite{20} general or is it an atypical set of experiments?

The answer to this question is that the process is quite general when we use a distribution as Eq. (6) for finite systems. The dependence of \( q \) on the number of particles already appeared in the literature \cite{5,19,21–24}.

The general scaling process discussed above shows in a direct way that \( q \) is a function of \( N \). Consider again the evolution of \( q \) under scaling as described by Fig. 1. Since a decimation process will reduce the number of sites \( N_s \), i.e., \( N_s \to N_s/2 \), one may expect that the number of particles in the system will be reduced as well. For a general case the law associating the number of sites will be not that described by Cordeiro (\( N \sim N_s^{3/2} \)); however, necessarily it will be a function of the number of sites, i.e., \( N = N(N_s) \). Consequently \( q = q(N_s) = q(N) \). For the region \( 1 < q < 2 \), \( q' < q \) and \( dq/dN > 0 \). Outside that region, i.e., for \( q < 1 \) or \( q > 2 \), the map on Fig. 1 shows that \( q' > q \) and consequently \( dq/dN < 0 \). However, in both situations as \( N \to \infty \) \( q \) deviates from 1. This is a quite important point for the discussion of scaling and entropy in the next section.

4. Scaling and entropy

We turn now our attention to the problem of scaling and entropy. The fact that \( q = q(N) \) poses a new problem for the Tsallis generalized entropy. The Tsallis \cite{6–9} definition of entropy is

\[
S_q = k_B \frac{1 - \int \rho_q(x) \, dx}{1 - q},
\]

where \( \rho_q(x) = n(x)/N \). Consider now two systems \( A \) and \( B \) with the entropy equation (12); the entropy for the system \( A + B \) will be given by \cite{6–9}

\[
S_{A+B} = S_A + S_B + (1 - q)S_A S_B.
\]
Since \( q \) changes with \( N \) an immediate question that arises is what \( q \) one shall use in Eq. (13), \( q_A \) or \( q_B \)? This question remains without an answer.

In order to obtain Eq. (13) Tsallis and collaborators have used the approximation

\[
\rho_{A+B} = \rho_A \rho_B .
\]  

(14)

This is the molecular chaos hypothesis [25]. In the Boltzmann–Gibbs statistics this result is valid only for the ideal gas, where energy and momenta are conserved in collisions. Consequently, we cannot expect that it will work for strongly correlated systems or granular materials. Moreover, simple mathematics shows that is impossible for a power law to satisfy Eq. (14), only an exponential will do it.

In order to show the approximative character of Eq. (13) let us consider the specific, and simplistic, situation where \( q \) is a fixed point of Eq. (8), i.e., \( q^* = f(q^*) \). We shall rule out the trivial point \( q_1^* = 1 \) and consider the point \( q_2^* = 2 \). We shall also consider as well that the systems \( A \) and \( B \) are equal. In this particular situation \( q \) does not depend on the number of sites. Let us consider another simplifying hypothesis that the number of particles and the number of sites are linearly connected. In this way let us call

\[
S_A = S_B = S(N, \beta) = 1 - \lambda ,
\]  

(15)

where \( \lambda = \sqrt{\beta/(16\pi^2)} < 1 \) in proper units. The last result was obtained using Eq. (12). Consequently using Eq. (13) we get

\[
S_{A+B} = 1 - \lambda^2 .
\]  

(16)

However, the parameters \( \beta \) of the distribution equation (7) changes with the scaling transformation (see Eq. (4)) as \( \beta \to \beta/4 \). As a consequence we have

\[
S(2N, \beta) = S(N, 4\beta) = 1 - 2\lambda \neq S_{A+B} .
\]  

(17)

Consequently, Eq. (17) agrees with Eq. (16) only for \( \lambda \to 0 \). This is the infinite diffusion limit \( \langle x^2 \rangle \to \infty \), which is a quite restrictive condition.

As already pointed by Fisher [26], the thermodynamical limit was not proved for systems where the interaction between two particles at distance \( r \), \( U(r) \propto r^{-\alpha} \), decays more slowly than the dimension \( d \) of the system \( (\alpha < d) \). A \( d > 1 \) gravitational system, is the best example of that. Consequently, we shall consider here two gravitational systems \( A \) and \( B \). If they are small, for example two diluted clouds, the deviation from Gaussian are not important since correlations will not affect strongly the system. Imagine that we start to increase the sizes of \( A \) and \( B \), in such a way that correlations becomes important. Now, if the distributions are given by Tsallians, they must move in a direction such that \( q \) becomes larger and larger than 1. Tsallians imply in \( q \) being a function which deviates from 1 as the system grows. That is a general conclusion. We shall notice that we did not make any comments on the validity or not of the Eq. (12).
5. Conclusion

We start with a decimation process for the RWD which brings us to an iterative equation similar to the Chapman–Kolmogorov equation [5]. We show the existence of some invariants and quasi-invariant of the equation. Those may be useful for the studies of generalized random walks. As a consequence we show that the parameter $q$ is an increasing function of the number of particles $N$, for $1 < q < 2$. Outside this interval one may expect an inverse behavior, i.e., since $q' = f(q) > q$, given by Eq. (8) will be larger than $q$ in this way, $q$ is a decreasing function of $N$.

We apply those concepts to the study of the scaling law related to the Tsallis entropy, and we made explicit the approximative character of the non-additivity as expressed by Tsallis and collaborators. We think that to clarify those questions is of fundamental importance to obtain a more general theory.

In the last years reasonable amount of work has been done in systems that present non-linear stability, such as fractures [27–29] and growth [30]. In general, a nucleation (see Refs. [31,32] and references therein) may occur when fluctuations grow beyond a non-returning point. Those fluctuations probably are not governed by Gaussians. As well, in granular materials and surface growth [30] inelastic collisions take place and the hypothesis of molecular chaos breaks down. For those we expect that the scaling concepts discussed here will be very important.

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References