A Unit Quaternion-based Spherical Trigonometry and a New Two-carrier Phase-quadrature Quaternion Modulation System

H.M. de Oliveira, D.R. de Oliveira, R.M. Campello de Souza

***Abstract*— A new Hyperspherical trigonometry based on Quaternion´s Algebra is introduced. Spatial curves on the surface of the unit-quaternion hypersphere are described in the parametric form. A draft of a new Continuous Quaternion Modulation System is presented, in which two modulation signals are used to drive paths in the normalized 4D-hypersphere. The outline of a PLL-based demodulation is also sketched.**

***Resumo*— Uma nova trigonometria hiperesférica baseada na Álgebra de Quatérnions é introduzida. Curvas espaciais sobre a superfície da hiperesfera unitária de quatérnions são descritas sob forma paramétrica. O esboço de um projeto para um novo sistema de Modulação Analógica com Quatérnions é apresentado, no qual dois sinais moduladores são utilizados para traçar caminhos na hiperesfera normalizada 4-D. Uma demodulação com base em PLL é apresentada.**

***Index Terms*— Quaternion´s algebra, hyperpherical trigonometry, quaternion modulation system.**

I. INTRODUCTION

A landmark on the modern algebra was achieved in 1843 by William Hamilton [1], who was the first to invent an algebra in which the commutative law of multiplication does not hold. The elements of such noncommutative algebra were called quaternions by Hamilton [2]. Since then, unit quaternions have provided a convenient mathematical notation for representing different spatial scenarios, rotations in three-dimensional space, topological curves and matrices.

A quaternion is a mathematical entity denoted by

$$\overleftarrow{Q} = c + x\breve{\imath} + y\breve{\jmath} + z\breve{k}, \tag{1}$$

with c, x, y, z: real numbers, and $\breve{\imath}$, $\breve{\jmath}$, $\breve{k}$: imaginary numbers.

Quaternions form a division ring, the elements of which can be viewed as a scalar plus a vector. They can be seen as some sort of hypercomplex numbers (that is, extended complex numbers). Equipped with the addition and multiplication of quaternions [3], they form a division ring. The addition of quaternions has an identity element: $0 = 0 + 0\breve{\imath} + 0\breve{\jmath} + 0\breve{k}$, and an inverse element: $-\overleftarrow{Q} = - c - x\breve{\imath} - y\breve{\jmath} - z\breve{k}$. The multiplication of quaternions has an identity element: $1 = 1+0\breve{\imath}+0\breve{\jmath}+0\breve{k}$. Fundamental formulae are:

$$\breve{\imath}^2=\breve{\jmath}^2=\breve{k}^2=\breve{\imath}\breve{\jmath}\breve{k}=-1. \tag{2}$$

Among many applications, besides quantum mechanics [4], the quaternions have been used in the coding of movements in a 3-D space [5] and in estimating the position and orientation of objects [6]. More related to this work, quaternion-based geodesic [7] and geometry of spherical curves [8] have also been introduced. There are also a loaded relationship among topological surfaces, codes and modulation [9-10]. We have particular interest in new special trigonometries as the trigonometry over finite fields recently introduced [11]. This paper presents the foundations of a "unit quaternion"-based spherical trigonometry. Given a point P on the surface of a unit sphere of coordinates (r,δ,λ), setting r=1, it can be associated with a unit quaternion in terms of the latitude and longitude according with

$$\overleftarrow{Q} := \sin(\delta).\cos(\lambda) + \cos(\delta).\sin(\lambda).\breve{\imath} + \sin(\delta).\sin(\lambda).\breve{\jmath} + \cos(\delta).\cos(\lambda).\breve{k}, \tag{3}$$

with these unit quaternions confined to be on the surface of a hypersphere of unity radius. Indeed, if n(.) denotes the magnitude of $\overleftarrow{Q}$ (norm of the quaternion), then $n(\overleftarrow{Q})=\|\overleftarrow{Q}\|=1$.

A matrix representation of quaternions can also be derived. In fact, one of the pioneers of the geometry of more than three dimensions, Arthur Cayley (1821-1895), offered matrices to represent noncommutative algebras. He was also able to link determinants with straight lines and planes,

$$\overleftarrow{Q} = \begin{bmatrix} \sin(\delta).\cos(\lambda) + j\cos(\delta).\sin(\lambda) & \sin(\delta).\sin(\lambda) + j\cos(\delta).\cos(\lambda) \\ -\sin(\delta).\sin(\lambda) + j\cos(\delta).\cos(\lambda) & \sin(\delta).\cos(\lambda) - j\cos(\delta).\sin(\lambda) \end{bmatrix} \tag{4}$$

Another keypoint is the inner product between two quaternions, which can be computed by

$$\begin{aligned} \overleftarrow{Q}_1.\overleftarrow{Q}_2 = {} & \sin(\delta_1).\sin(\delta_2).\cos(\lambda_1).\cos(\lambda_2) + \\ & +\cos(\delta_1).\cos(\delta_2).\sin(\lambda_1).\sin(\lambda_2) + \\ & +\sin(\delta_1).\sin(\delta_2).\sin(\lambda_1).\sin(\lambda_2) + \\ & +\cos(\delta_1).\cos(\delta_2).\cos(\lambda_1).\cos(\lambda_2). \end{aligned} \tag{5}$$

Rearranging this expression, we get the symmetric and nice-looking expression:

$$\overleftarrow{Q}_1.\overleftarrow{Q}_2 = \cos(\Delta\delta).\cos(\Delta\lambda), \tag{6}$$

where $\Delta\delta := \delta_2 - \delta_1$ and $\Delta\lambda := \lambda_2 - \lambda_1$.

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This relationship is much more symmetric than the conventional inner product over the Euclidean space $\mathbb{R}^3$, derived using spherical coordinates. The distance between two unit quaternions on a sphere of radius R can be computed as $d:=R.\mathrm{arc}(\overleftarrow{Q}_1\overleftarrow{Q}_2)$, and the angle is computed using the dot product

$$\mathrm{arc}(\overleftarrow{Q}_1\overleftarrow{Q}_2):=\arccos(\overleftarrow{Q}_1.\overleftarrow{Q}_2),\ \text{since}\ |\overleftarrow{Q}_1|=|\overleftarrow{Q}_2|=1. \quad (7)$$

Therefore,

$$\hat{d}= R.\ \arccos\ (\cos(\Delta\delta).\cos(\Delta\lambda)). \quad (8)$$

JavaScript and most current computer languages use IEEE 754 64-bit floating-point numbers, which provide 15 significant figures of precision. In the following, we present a Javascript to evaluate the distance of points of known latitude and longitude, assuming a radius equals to the radius of the Earth.

JavaScript:

```
var R = 6371; // km
var d = Math.acos(Math.cos(lat2-lat1)*Math. *
Math.cos(lon2-lon1))*R;
```

The complete script is available to run at the URL http://www2.ee.ufpe.br/codec/quaternion_distance.jar

2. A Spherical Trigonometric over Quaternions

In order to define a hyper-trigonometry, let us take the identity element as a reference point (λ=0 and $\delta=\pi/2$) and define the functions in terms of the angle between this reference and an arbitrary unity quaternion ($\overleftarrow{Q}$). Given a quaternion (a point on the surface of the 4D-hypersphere), the following trigonometric functions are promptly defined

$$\cos(\overleftarrow{Q})=\cos(\lambda).\sin(\delta) \quad (9a)$$
$$\sin(\overleftarrow{Q})=\sqrt{1-\cos^2(\lambda).\sin^2(\delta)} \quad (9b)$$
$$\tan(\overleftarrow{Q})=\sqrt{sec^2(\lambda).cossec^2(\delta)-1} \quad (9c)$$
$$\sec(\overleftarrow{Q})=\sec(\lambda).\mathrm{cossec}(\delta). \quad (9d)$$

For instance, for the particular point $\overleftarrow{P}$=UFPE with coordinates λ=lon=-34.953604° and δ=lat=-8.052771°, we have: cos(UFPE)=0.18219; sin(UFPE)=0.98326; tan(UFPE)= 0.18529.

Now, setting $\delta=\pi/2$, λ arbitrary, the trigonometry is confined to the X-Y plane and collapse to the ordinary trigonometry:

$$\cos(\overleftarrow{Q})=\cos(\lambda),\ \sin(\overleftarrow{Q})=\sqrt{1-cos^2(\lambda)};\ \tan^2(\overleftarrow{Q})=\sec^2(\lambda)-1$$
and $\sec(\overleftarrow{Q})=\sec(\lambda)$, as expected.

When dealing with a sphere of radius R, the quaternion is defined merely by $R\overleftarrow{Q}$. There is a close relationship between this quaternion-distance and the 3D-Euclidean distance of the geodesic path defined by the two coordinates. Let us consider the point described by the system illustrated in Fig. 1. By using the properties of spherical triangles, the distance between two points P_1P_2can be computed by the well-known relation:

$$\cos(\widehat{P_1P_2})=\cos(\alpha)=\cos(\delta_1).\cos(\delta_2).\cos(\Delta\lambda)+\sin(\delta_1).\sin(\delta_2)$$
$$\text{so that } d=R.\alpha. \quad (10)$$

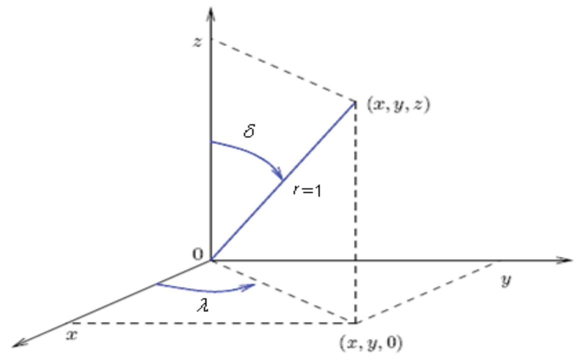

Figure 1. Standard geo-coordinates for a unit Sphere.

Considering:
P1: RECIFE @ 8°03′14′′S 34°57′13′′W (lat=-8.05277, lon=-34.95360),
P2: PARIS @ 48°49′36′′N 2°20′41′′E (lat=48.82668, lon= 2.34649),
the distance computed using Eq(10) is exactly the Euclidian distance used in GPS (d=7,295 km). The use of this formula allows the calculation of great-circle distances between the two points – that is, the shortest distance over the earth's surface, ignoring possible hills [12]. In contrast, the distance measured by Eq.(8) is carried out using another inner product ($\hat{d}$=7,143 km). However, close estimations can be obtained, especially when the latitude of the two points are not very different. If the Euclidian distance is required, a correction factor can be used,

$$\alpha=\arccos(\cos(\Delta\delta).\cos(\Delta\lambda)+\varepsilon), \quad (11)$$

where ε is a correction factor given by: $\varepsilon:=(1-\cos(\Delta\lambda)).\sin(\delta_1).\sin(\delta_2)$. For a much accurate result, one can replace R by R=(6,371.005076123 + mean altitude of the two points) km.

Additional improvements can easily be made by considering ellipsoidal-shaped quaternions instead of spherical ones. In this case, we redefine Eq.(3), in terms of the flattening f and the major axis a of the ellipsoid, as

$$\overleftarrow{Q}:=a.\sin(\delta).\cos(\lambda)+a.(1-f).\cos(\delta).\sin(\lambda).\breve{\imath}+$$
$$+a.\sin(\delta).\sin(\lambda).\breve{\jmath}+a.(1-f)\cos(\delta).\cos(\lambda).\breve{k}. \quad (12)$$

To be in compliance with the Brazilian Geodesic System [13], we adopt a=6,378.160 km (major semi-axis) and f=1/298.25 (flattening). When using the international ellipsoid model of Hayford, a=6,378.38800 km and f=1/297.0 [13].

The inner product given by Eq.(6) is then replaced by:

$$\overleftarrow{Q}_1.\overleftarrow{Q}_2 = a^2.\cos(\Delta\lambda).\{\cos(\delta_1).\cos(\delta_2).(1-f)^2+$$
$$+\sin(\delta_1).\sin(\delta_2)\}, \quad (13)$$

which reduces to Eq.(6) when a=R (R=1) and f=0.

Further geometrical considerations are presented in the sequel. If every point on the a spherical shell is fully described by a quaternion, then any trajectory can be described in parametrized form, that is

$$r(t)=q_1(t)+q_2(t)\breve{\imath}+q_3(t)\breve{\jmath}+q_4(t)\breve{k}. \quad (14)$$

This approach allows the use of vector functions for parametrizing a curve C, as illustrate in Figure 2.

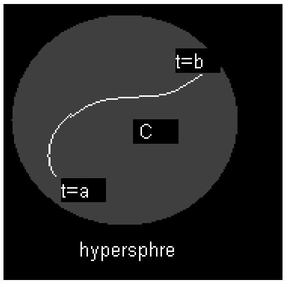

Figure 2. A trajectory C (spatial curve) on the surface of the quaternion hypersphere. The curve is oriented and starts when $t=a<b$ and finishes when the elapsed time is b-a.

Any trajectory (a spatial curve) can be modeled through vector functions according with:

$$r(t):=\cos(\lambda(t)).\sin(\delta(t))+\cos(\delta(t)).\sin(\lambda(t)).\breve{i}+ \\ +\sin(\delta(t)).\sin(\lambda(t)).\breve{j}+\cos(\delta(t)).\cos(\lambda(t)).\breve{k}\,, \quad (15)$$

where $r(t)$ are the parametric equations of any curve C.

Indeed, the length L of a surface curve on the hypersphere can be computed by

$$L=\int_a^b\sqrt{\sum_{i=1}^4\left(\frac{dq_i(t)}{dt}\right)^2}\;dt, \quad (16)$$

where $q_i(t)$ are continuous and differentiable functions. For instance, let us compute the length of the geodesic path at the surface of a hypersphere of radius R= 6,371 km from the quaternion δ_1=53.14722^o and λ_1=-1.84944^o to δ_2=20.20444^o and λ_2=-8.97389^o.

From t=0 to t=48h, at constant speeds v_δ=-0.012 rad/h and v_λ=-0.00259 rad/h, a computation using Mathcad™ furnished roughly 3,748 km. The evaluation of the distance using Eq.(8-10) furnished 3,738 km (<0.27%).

What is a possible relationship between the tools described in this paper and the Telecommunications field? That is a question that certainly was made through the reading. Not only first approximation GPS-issues are potentially concerned with, but mainly new modulation schemes can be devised. We are currently examining a two-carrier phase-quadrature continuous modulation, in which two analog signals are used to pilot $\lambda(t)$ and $\delta(t)$, so as TO describe a particular surface curve in the 4D-hypersphere. A low carrier frequency w_m is used in an FM modulator that drives the unit-quaternion coordinates δ and λ.

3. A Draft of a Continuous Quaternion Modulation System

We propose here a two-carrier phase-quadrature quaternion modulation system as shown in the block diagram of Fig. 3. In this system, two analog information signals are used to pilot the angles $\lambda(t)$ and $\delta(t)$ so as to describe a particular surface curve in the quaternion sphere,

$$\delta(t)\coloneqq w_m t+2\pi K_{f1}\int_{-\infty}^t f_I(t')dt', \quad (17)$$

$$\lambda(t)\coloneqq w_m t+2\pi K_{f2}\int_{-\infty}^t f_Q(t')dt'. \quad (18)$$

Two FM modulators using the direct method [14] are provided to generate the basic signal with quadrature carriers. This is illustrated in Figures 3a and 3b.

The output of the four FM systems are $\cos(\lambda(t))$, $\sin(\lambda(t))$ and $\cos(\delta(t))$, $\sin(\delta(t))$, respectively. Then, four balanced four-quadrant modulator (Gilbert-cells type [14]), combine the basic signals so as to compute the (quaternion) coordinates $\{q_i(t)\}$ of the vector function that describes the surface curve. In order to transmit such quaternion coordinates, two multiplex (in phase and in quadrature multiplex, Fig. 3c) can be used so as to require only a couple of high frequency (w_{c1}, w_{c2}) carriers.

A straightforward option for this design could be a standard FDM system to independently transmit the quaternion components. If a four-carrier FDM system is used two transmit information of the quaternion, a passband filter bank centered at the frequency w_{ci} i=1,2…4 allows demodulating the envelope, recovering then the i^{th}-coordinate of the quaternion, i=1,2…4. In such an approach, the synchronous carrier detection required by using the two “quadrature multiplex” is no more required, which can be an advantage in some communication scenarios. An OFDM-based system could as well be designed to carry the quaternion information.

This kind of communication system can be interpreted as some sort of modulation associated with twisted signal locus, as described in the seminal book by Wozencraft-Jacobs [15, p.613].

If digital information is available at the input as an alternative for the two analog signals, then CPFSK (Continuous-Phase Frequency-Shift Keying) can be used instead of FM modulators.

The demodulator circuit first performs a synchronous detection of the two carriers (a passband filter is used) and the quaternion coordinates $q_1(t)$ $q_3(t)$ and $q_2(t)$ $q_4(t)$ are retrieved. A limiter circuit should be used, since there is no information in the amplitude. Then, four squaring circuits are used and their output are combined to derive both signals $\sin(\delta(t))$ and $\cos(\delta(t))$. The key quaternion coordinate retrieval relationships are:

$$q_1^2(t)+q_3^2(t)=\sin^2(\delta(t)), \quad (19a)$$
$$q_2^2(t)+q_4^2(t)=\cos^2(\delta(t)), \quad (19b)$$
$$q_2^2(t)+q_3^2(t)=\sin^2(\lambda(t)), \quad (19c)$$
$$q_1^2(t)+q_4^2(t)=\cos^2(\lambda(t)). \quad (19d)$$

Therefore, there exists some *analog redundancy* in the scheme that can be exploited; $q_1(t),\ldots,q_4(t)$ work as a kind of analog parity-check equations. These signals undertake a final demodulation step (Figs. 5-6), which is achieved with a PLL as an FM demodulator [14], now tuned at low free-frequency, leading to a guesstimate of the signals $f_I(t)$ and $f_Q(t)$. From the double arc identities, both $\sin^2(.)$ and $\cos^2(.)$, in Eqs. (19), can be expressed in terms of $\cos(2\delta(t))$ or $\cos(2\lambda(t))$. Accordingly, the free frequency of each PLL is $2w_m \ll w_c$, i.e., twice the basic low frequency. Naturally, we assume that such a frequency is greater than the maximum frequency of both the bandlimited input signals f_I and f_Q and it is enough to avoid spectrum overlap. The demodulation of $\lambda(t)$ is carried out in a similar way. There are different ways to combine the two redundant estimations, but this point is not addressed here in this paper.

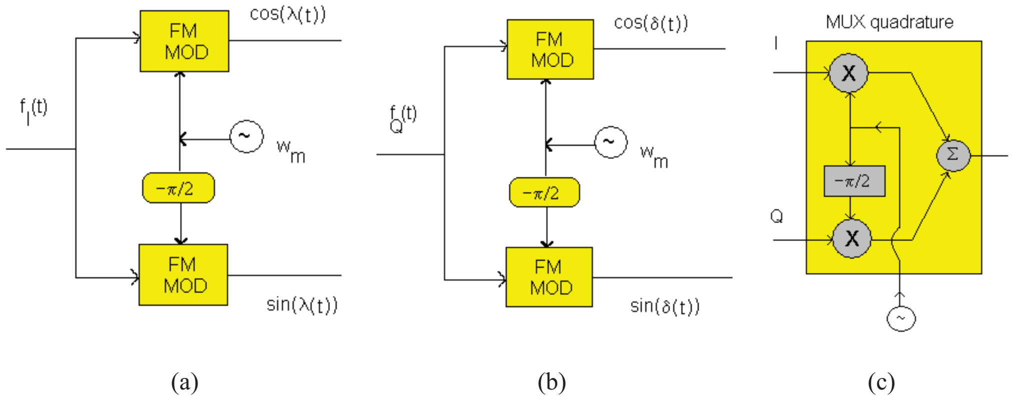

Figure 3. Block diagram of circuits for generating quaternion coordinates: a) *in phase* signal drives longitude, b) *in quadrature* signal drives latitude, c) A two-signal basic multiplex used to convey the quaternion coordinates over the same carrier.

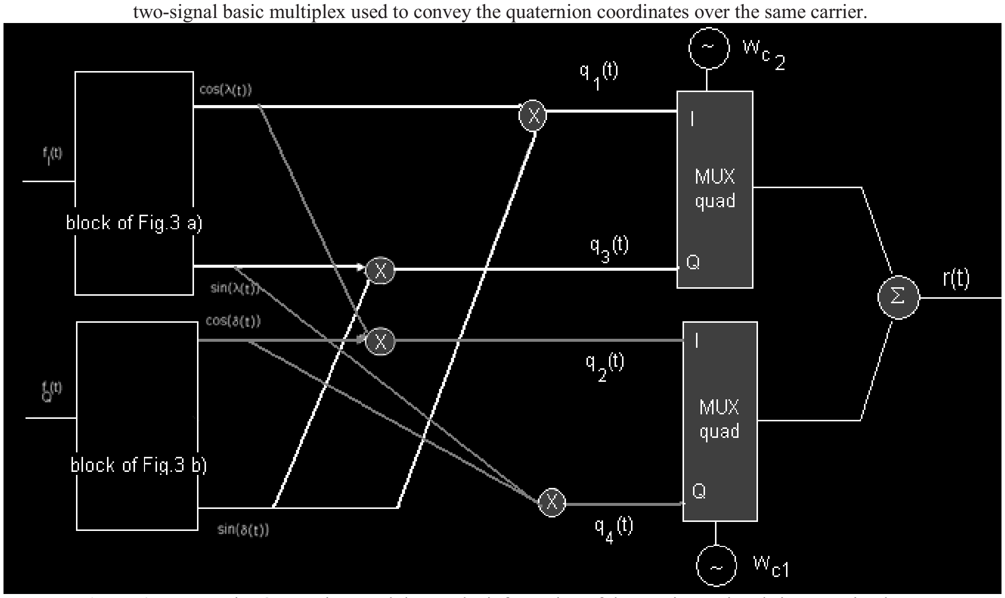

Figure 4. Two-carrier Quaternion Modulator. The information of the two input signals is transmitted as a twisted-locus $r(t)$ of a surface curve on the quaternion hypersphere given by Eq.(15).

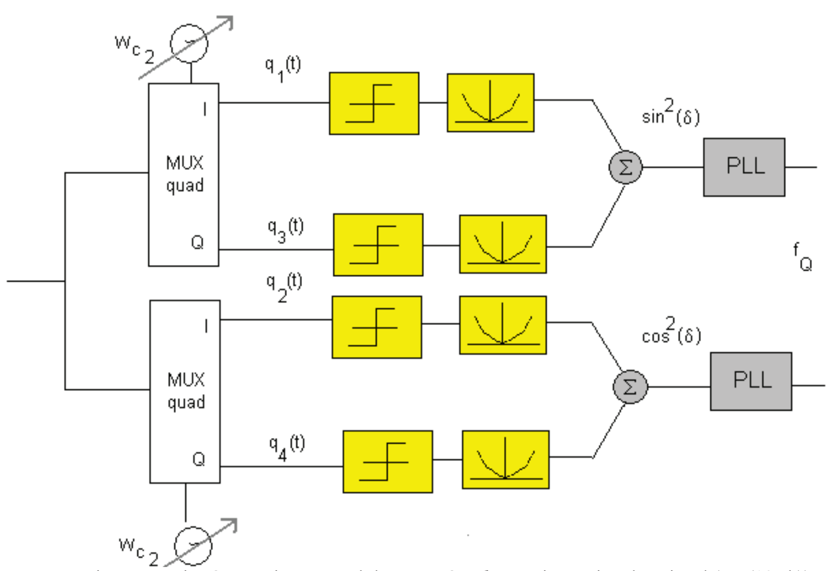

Figure 5. Basic two-carrier Quaternion Demodulator: *In Quadrature* input signal retrieval (Eq.(19a,b)).

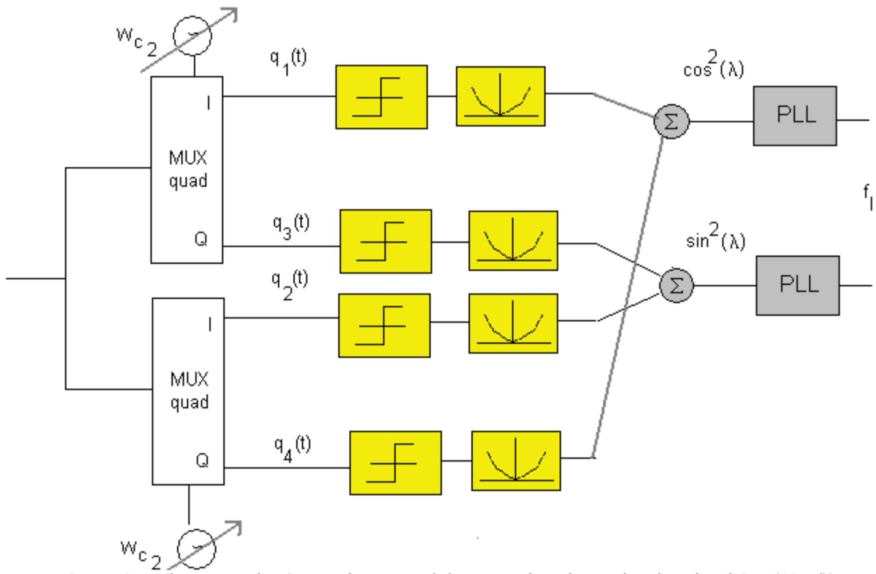

Figure 6. Basic two-carrier Quaternion Demodulator: *In phase* input signal retrieval (Eq.(19c,d)).

Another point that deserves a careful investigation is the bandwidth requirements of these systems. At a glance, the bandwidth per carrier B is roughly:

$$B = B_{\text{FM1}} + B_{\text{FM2}}, \text{ Hz/carrier}, \qquad (20)$$

where B_{FM1}, B_{FM2} are the bandwidth needed for the FM-signals $\cos(\lambda(t))$, $\cos(\delta(t))$, respectively, estimated by Carson's rule. This is twice the band of one FM signal if identical modulators are used. Also, the analysis and performance assessing in the presence of Gaussian noise are currently under examination, to gain insight into the behavior of the proposed system.

4. Concluding Remarks

This paper presents some preliminary investigation on potential applications of quaternions. In particular, a new hyperspherical trigonometry based on quaternion´s algebra is introduced and the description of surface curves on the unit-quaternion sphere is addressed. Parametrizing is also an important issue for quantum coding [16]. One application of these tools is the design of new twisted-locus analog communication systems, where the modulation scheme is quaternion-based. This approach also provides analog redundancy, which is a promising aspect of the technique. Albeit the performance results have not be assessed yet, the true nature of these schemes is quite new and they extensively make use of FM modulators and PLL, which are known to provide good noise immunity properties.

References

[1] C.C. MacDuffee, Algebra's debt to Hamilton, *Scripta Mathematica*, vol.**10**, pp.25-36, 1944.

[2] B.L. van der Waerden, Hamilton´s Discovery of Quaternions, *Mathematical Magazine*, JSTOR, vol.**49**,N5,Nov., pp.227-234, 1976. Reprinted in: G.L. Alexanderson, P. Ross (Eds.), *The Harmony of the world: 75 years of Mathematics magazine*, The Mathematical Association of America, pp.143-149, 2007.

[3] F. Zhang, Quaternions and Matrices of Quaternions, *Linear Algebra and its Applications*, vol.**251**, pp.21-57, 1997. doi> 10.1016/0024-3795(95)00543-9

[4] S. De Leo, W.A. Rodrigues, Quantum mechanics: From complex to complexified quaternions, *Int. J. of Theoretical Physics*, vol.**36**, N.12, pp.2725-2757, 1997. doi> 10.1007/BF02435708

[5] P.-G. Maillot, Using Quaternions for Coding 3-D Transformations, in *Graphic Gems* (Ed. J. Arvo), Academic Press, NY,pp.498-515, 1990.

[6] M.W. Walker, L. Shao, R.A. Volz, Estimating 3-D Location Parameters Using Dual Number Quaternions, CVGIP: *Image Understanding*. **54**, N.3, Nov., pp. 358-367, 1991. doi> 10.1016/1049-9660(91)90036-O

[7] F. Catoni, P. Zanpetti, R. Cannata, L. Bordoni, Geodesic in Hypercomplex Number Systems, Application to Commutative Quaternions, NASA 19880210463, Italy, 1997. Also *Advances in Applied Clifford Algebras*,vol.**16**, N.1, pp.1-28, 2006. doi> http://dx.doi.org/10.1007/s00006-006-0002-y

[8] The Geometry of Spherical Curves and the Algebra of Quaternions, *Russian Mathematical Surveys*, vol.**50**, N.1, pp.1-68, 1995. doi> 10.1070/RM1995v050n01ABEH001662.

[9] C. D. Albuquerque, R. Palazzo, and E. B. Silva, Topological quantum codes on compact surfaces with genus g ≥ 2, *J. Math. Phys*., vol.**50**, N.2, pp. 023513-023513-20, 2009. doi> 10.1063/1.3081056

[10] J. D. Lima, Luana P. R. C. Lima, Topological Projects of Modulations on Surfaces, *Annals of the XXXI Congresso Nacional de Matemática Aplicada e Computacional*. Belém - PA, 8 a 11 de setembro de 2008. Available at ST7 http://www.sbmac.org.br/publi_cnmac/linux/list7.html pp.1-5.

[11] R.M. Campello de Souza, H.M. de Oliveira, M.M.C. de Souza and L. B. E. Palma, Hartley Number-Theoretic Transforms, *Proceedings of the IEEE International Symposium on Information Theory*, Washington DC, p. 210, June 2001. doi> 0.1007/BF02435708

[12] http//:www.chemical-ecology.net/java/lat-long.htm (Feb. 2011).

[13] J. F. G. Monico, *Posicionamento pelo NAVSTAR-GPS*: descrição, fundamentos e aplicações, Editora Unesp, (287p.), 2000.

[14] P.H. Young, *Electronic Communication Techniques*, 5th Edition, Prentice Hall, (960p.), 2003.

[15] J.M. Wozencraft, I.M. Jacobs, *Principles of Communication Engineering*, Wiley, 1965.

[16] R. Mosseri and R. Dandoloff , Geometry of entangled states, block spheres and Hopf fibrations, *J. Phys. A: Math. Gen.* **34** pp. 10243–52 2001.