

The Ideal Mathematician

WE WILL CONSTRUCT a portrait of the “ideal mathematician.” By this we do not mean the perfect mathematician, the mathematician without defect or limitation. Rather, we mean to describe the most mathematician-like mathematician, as one might describe the ideal thoroughbred greyhound, or the ideal thirteenth-century monk. We will try to construct an impossibly pure specimen, in order to exhibit the paradoxical and problematical aspects of the mathematician’s role. In particular, we want to display clearly the discrepancy between the actual work and activity of the mathematician and his own perception of his work and activity.

The ideal mathematician’s work is intelligible only to a small group of specialists, numbering a few dozen or at most a few hundred. This group has existed only for a few decades, and there is every possibility that it may become extinct in another few decades. However, the mathematician regards his work as part of the very structure of the world, containing truths which are valid forever, from the beginning of time, even in the most remote corner of the universe.

He rests his faith on rigorous proof; he believes that the difference between a correct proof and an incorrect one is an unmistakable and decisive difference. He can think of no condemnation more damning than to say of a student, “He doesn’t even know what a proof is.” Yet he is able to give no coherent explanation of what is meant by rigor, or what is required to make a proof rigorous. In his own work, the line between complete and incomplete proof is always somewhat fuzzy, and often controversial.

To talk about the ideal mathematician at all, we must have a name for his “field,” his subject. Let’s call it, for instance, “non-Riemannian hypersquares.”

He is labeled by his field, by how much he publishes, and especially by whose work he uses, and by whose taste he follows in his choice of problems.

He studies objects whose existence is unsuspected by all except a handful of his fellows. Indeed, if one who is not an initiate asks him what he studies, he is incapable of showing or telling what it is. It is necessary to go through an arduous apprenticeship of several years to understand the theory to which he is devoted. Only then would one's mind be prepared to receive his explanation of what he is studying. Short of that, one could be given a "definition," which would be so recondite as to defeat all attempts at comprehension.

The objects which our mathematician studies were unknown before the twentieth century; most likely, they were unknown even thirty years ago. Today they are the chief interest in life for a few dozen (at most, a few hundred) of his comrades. He and his comrades do not doubt, however, that non-Riemannian hypersquares have a real existence as definite and objective as that of the Rock of Gibraltar or Halley's comet. In fact, the proof of the existence of non-Riemannian hypersquares is one of their main achievements, whereas the existence of the Rock of Gibraltar is very probable, but not rigorously proved.

It has never occurred to him to question what the word "exist" means here. One could try to discover its meaning by watching him at work and observing what the word "exist" signifies operationally.

In any case, for him the non-Riemannian hypersquare exists, and he pursues it with passionate devotion. He spends all his days in contemplating it. His life is successful to the extent that he can discover new facts about it.

He finds it difficult to establish meaningful conversation with that large portion of humanity that has never heard of a non-Riemannian hypersquare. This creates grave difficulties for him; there are two colleagues in his department who know something about non-Riemannian hypersquares, but one of them is on sabbatical, and the other is much more interested in non-Eulerian semirings. He goes to conferences, and on summer visits to colleagues, to meet

people who talk his language, who can appreciate his work and whose recognition, approval, and admiration are the only meaningful rewards he can ever hope for.

At the conferences, the principal topic is usually "the decision problem" (or perhaps "the construction problem" or "the classification problem") for non-Riemannian hypersquares. This problem was first stated by Professor Nameless, the founder of the theory of non-Riemannian hypersquares. It is important because Professor Nameless stated it and gave a partial solution which, unfortunately, no one but Professor Nameless was ever able to understand. Since Professor Nameless' day, all the best non-Riemannian hypersquarers have worked on the problem, obtaining many partial results. Thus the problem has acquired great prestige.

Our hero often dreams he has solved it. He has twice convinced himself during waking hours that he had solved it but, both times, a gap in his reasoning was discovered by other non-Riemannian devotees, and the problem remains open. In the meantime, he continues to discover new and interesting facts about the non-Riemannian hypersquares. To his fellow experts, he communicates these results in a casual shorthand. "If you apply a tangential mollifier to the left quasi-martingale, you can get an estimate better than quadratic, so the convergence in the Bergstein theorem turns out to be of the same order as the degree of approximation in the Steinberg theorem."

This breezy style is not to be found in his published writings. There he piles up formalism on top of formalism. Three pages of definitions are followed by seven lemmas and, finally, a theorem whose hypotheses take half a page to state, while its proof reduces essentially to "Apply Lemmas 1-7 to definitions A-H."

His writing follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being. It gives the impression that, from the stated definitions, the desired results follow infallibly by a purely mechanical procedure. In fact, no computing machine has ever been built that could accept his definitions as inputs. To read his proofs, one must be privy to a whole subcul-

ture of motivations, standard arguments and examples, habits of thought and agreed-upon modes of reasoning. The intended readers (all twelve of them) can decode the formal presentation, detect the new idea hidden in lemma 4, ignore the routine and uninteresting calculations of lemmas 1, 2, 3, 5, 6, 7, and see what the author is doing and why he does it. But for the noninitiate, this is a cipher that will never yield its secret. If (heaven forbid) the fraternity of non-Riemannian hypersquarers should ever die out, our hero's writings would become less translatable than those of the Maya.

The difficulties of communication emerged vividly when the ideal mathematician received a visit from a public information officer of the University.

P.I.O. I appreciate your taking time to talk to me. Mathematics was always my worst subject.

I.M.: That's O.K. You've got your job to do.

P.I.O. I was given the assignment of writing a press release about the renewal of your grant. The usual thing would be a one-sentence item, "Professor X received a grant of Y dollars to continue his research on the decision problem for non-Riemannian hypersquares." But I thought it would be a good challenge for me to try and give people a better idea about what your work really involves. First of all, what is a hypersquare?

I.M. I hate to say this, but the truth is, if I told you what it is, you would think I was trying to put you down and make you feel stupid. The definition is really somewhat technical, and it just wouldn't mean anything at all to most people.

P.I.O. Would it be something engineers or physicists would know about?

I.M. No. Well, maybe a few theoretical physicists. Very few.

P.I.O. Even if you can't give me the real definition, can't you give me some idea of the general nature and purpose of your work?

Varieties of Mathematical Experience

- I.M.:* All right, I'll try. Consider a smooth function f on a measure space Ω taking its value in a sheaf of germs equipped with a convergence structure of saturated type. In the simplest case . . .
- P.I.O.:* Perhaps I'm asking the wrong questions. Can you tell me something about the applications of your research?
- I.M.:* Applications?
- P.I.O.:* Yes, applications.
- I.M.:* I've been told that some attempts have been made to use non-Riemannian hypersquares as models for elementary particles in nuclear physics. I don't know if any progress was made.
- P.I.O.:* Have there been any major breakthroughs recently in your area? Any exciting new results that people are talking about?
- I.M.:* Sure, there's the Steinberg-Bergstein paper. That's the biggest advance in at least five years.
- P.I.O.:* What did they do?
- I.M.:* I can't tell you.
- P.I.O.:* I see. Do you feel there is adequate support in research in your field?
- I.M.:* Adequate? It's hardly lip service. Some of the best young people in the field are being denied research support. I have no doubt that with extra support we could be making much more rapid progress on the decision problem.
- P.I.O.:* Do you see any way that the work in your area could lead to anything that would be understandable to the ordinary citizen of this country?
- I.M.:* No.
- P.I.O.:* How about engineers or scientists?
- I.M.:* I doubt it very much.
- P.I.O.:* Among pure mathematicians, would the majority be interested in or acquainted with your work?
- I.M.:* No, it would be a small minority.
- P.I.O.:* Is there anything at all that you would like to say about your work?
- I.M.:* Just the usual one sentence will be fine.

P.I.O.: Don't you want the public to sympathize with your work and support it?

I.M.: Sure, but not if it means debasing myself.

P.I.O.: Debasing yourself?

I.M.: Getting involved in public relations gimmicks, that sort of thing.

P.I.O.: I see. Well, thanks again for your time.

I.M.: That's O.K. You've got a job to do.

Well, a public relations officer. What can one expect? Let's see how our ideal mathematician made out with a student who came to him with a strange question.

Student: Sir, what is a mathematical proof?

I.M.: You don't know *that*? What year are you in?

Student: Third-year graduate.

I.M.: Incredible! A proof is what you've been watching me do at the board three times a week for three years! That's what a proof is.

Student: Sorry, sir, I should have explained. I'm in philosophy, not math. I've never taken your course.

I.M.: Oh! Well, in that case—you have taken *some* math, haven't you? You know the proof of the fundamental theorem of calculus—or the fundamental theorem of algebra?

Student: I've seen arguments in geometry and algebra and calculus that were called proofs. What I'm asking you for isn't *examples* of proof, it's a definition of proof. Otherwise, how can I tell what examples are correct?

I.M.: Well, this whole thing was cleared up by the logician Tarski, I guess, and some others, maybe Russell or Peano. Anyhow, what you do is, you write down the axioms of your theory in a formal language with a given list of symbols or alphabet. Then you write down the hypothesis of your theorem in the same symbolism. Then you show that you can transform the hypothesis step by step, using the rules of logic, till you get the conclusion. That's a proof.

Student: Really? That's amazing! I've taken elementary



Alfred Tarski
1902 -

Varieties of Mathematical Experience

- and advanced calculus, basic algebra, and topology, and I've never seen that done.
- I.M.:* Oh, of course no one ever really *does* it. It would take forever! You just show that you could do it, that's sufficient.
- Student:* But even that doesn't sound like what was done in my courses and textbooks. So mathematicians don't really do proofs, after all.
- I.M.:* Of course we do! If a theorem isn't proved, it's nothing.
- Student:* Then what is a proof? If it's this thing with a formal language and transforming formulas, nobody ever proves anything. Do you have to know all about formal languages and formal logic before you can do a mathematical proof?
- I.M.:* Of course not! The less you know, the better. That stuff is all abstract nonsense anyway.
- Student:* Then really what *is* a proof?
- I.M.:* Well, it's an argument that convinces someone who knows the subject.
- Student:* Someone who knows the subject? Then the definition of proof is subjective; it depends on particular persons. Before I can decide if something is a proof, I have to decide who the experts are. What does that have to do with proving things?
- I.M.:* No, no. There's nothing subjective about it! Everybody knows what a proof is. Just read some books, take courses from a competent mathematician, and you'll catch on.
- Student:* Are you sure?
- I.M.:* Well—it is possible that you won't, if you don't have any aptitude for it. That can happen, too.
- Student:* Then *you* decide what a proof is, and if I don't learn to decide in the same way, you decide I don't have any aptitude.
- I.M.:* If not me, then who?

Then the ideal mathematician met a positivist philosopher.

- P.P.:* This Platonism of yours is rather incredible. The silliest undergraduate knows enough not to multiply entities, and here you've got not just a handful, you've got them in uncountable infinities! And nobody knows about them but you and your pals! Who do you think you're kidding?
- I.M.:* I'm not interested in philosophy, I'm a mathematician.
- P.P.:* You're as bad as that character in Molière who didn't know he was talking prose! You've been committing philosophical nonsense with your "rigorous proofs of existence." Don't you know that what exists has to be observed, or at least observable?
- I.M.:* Look, I don't have time to get into philosophical controversies. Frankly, I doubt that you people know what you're talking about; otherwise you could state it in a precise form so that I could understand it and check your argument. As far as my being a Platonist, that's just a handy figure of speech. I never thought hypersquares existed. When I say they do, all I mean is that the axioms for a hypersquare possess a model. In other words, no formal contradiction can be deduced from them, and so, in the normal mathematical fashion, we are free to postulate their existence. The whole thing doesn't really mean anything, it's just a game, like chess, that we play with axioms and rules of inference.
- P.P.:* Well, I didn't mean to be too hard on you. I'm sure it helps you in your research to imagine you're talking about something real.
- I.M.:* I'm not a philosopher, philosophy bores me. You argue, argue and never get anywhere. My job is to prove theorems, not to worry about what they mean.

The ideal mathematician feels prepared, if the occasion should arise, to meet an extragalactic intelligence. His first effort to communicate would be to write down (or other-

wise transmit) the first few hundred digits in the binary expansion of pi. He regards it as obvious that any intelligence capable of intergalactic communication would be mathematical and that it makes sense to talk about mathematical intelligence apart from the thoughts and actions of human beings. Moreover, he regards it as obvious that binary representation and the real number pi are both part of the intrinsic order of the universe.

He will admit that neither of them is a natural object, but he will insist that they are discovered, not invented. Their discovery, in something like the form in which we know them, is inevitable if one rises far enough above the primordial slime to communicate with other galaxies (or even with other solar systems).

The following dialogue once took place between the ideal mathematician and a skeptical classicist.

S.C.: You believe in your numbers and curves just as Christian missionaries believed in their crucifixes. If a missionary had gone to the moon in 1500, he would have been waving his crucifix to show the moon-men that he was a Christian, and expecting them to have their own symbol to wave back.* You're even more arrogant about your expansion of pi.

I.M.: Arrogant? It's been checked and rechecked, to 100,000 places!

S.C.: I've seen how little you have to say even to an American mathematician who doesn't know your game with hypersquares. You don't get to first base trying to communicate with a theoretical physicist; you can't read his papers any more than he can

* Cf. the description of Coronado's expedition to Cibola, in 1540: ". . . there were about eighty horsemen in the vanguard besides twenty-five or thirty foot and a large number of Indian allies. In the party went all the priests, since none of them wished to remain behind with the army. It was their part to deal with the friendly Indians whom they might encounter, and they especially were bearers of the Cross, a symbol which . . . had already come to exert an influence over the natives on the way" (H. E. Bolton, *Coronado*, University of New Mexico Press, 1949).

read yours. The research papers in your own field written before 1910 are as dead to you as Tutankhamen's will. What reason in the world is there to think that you could communicate with an extragalactic intelligence?

I.M.: If not me, then who else?

S.C.: Anybody else! Wouldn't life and death, love and hate, joy and despair be messages more likely to be universal than a dry pedantic formula that nobody but you and a few hundred of your type will know from a hen-scratch in a farmyard?

I.M.: The reason that my formulas are appropriate for intergalactic communication is the same reason they are not very suitable for terrestrial communication. Their content is not earthbound. It is free of the specifically human.

S.C.: I don't suppose the missionary would have said quite that about his crucifix, but probably something rather close, and certainly no less absurd and pretentious.

The foregoing sketches are not meant to be malicious; indeed, they would apply to the present authors. But it is a too obvious and therefore easily forgotten fact that mathematical work, which, no doubt as a result of long familiarity, the mathematician takes for granted, is a mysterious, almost inexplicable phenomenon from the point of view of the outsider. In this case, the outsider could be a layman, a fellow academic, or even a scientist who uses mathematics in his own work.

The mathematician usually assumes that his own view of himself is the only one that need be considered. Would we allow the same claim to any other esoteric fraternity? Or would a dispassionate description of its activities by an observant, informed outsider be more reliable than that of a participant who may be incapable of noticing, not to say questioning, the beliefs of his coterie?

Mathematicians know that they are studying an objective reality. To an outsider, they seem to be engaged in an esoteric communion with themselves and a small clique of

friends. How could we as mathematicians prove to a skeptical outsider that our theorems have meaning in the world outside our own fraternity?

If such a person accepts our discipline, and goes through two or three years of graduate study in mathematics, he absorbs our way of thinking, and is no longer the critical outsider he once was. In the same way, a critic of Scientology who underwent several years of "study" under "recognized authorities" in Scientology might well emerge a believer instead of a critic.

If the student is unable to absorb our way of thinking, we flunk him out, of course. If he gets through our obstacle course and then decides that our arguments are unclear or incorrect, we dismiss him as a crank, crackpot, or misfit.

Of course, none of this proves that we are not correct in our self-perception that we have a reliable method for discovering objective truths. But we must pause to realize that, outside our coterie, much of what we do is incomprehensible. There is no way we could convince a self-confident skeptic that the things we are talking about make sense, let alone "exist."

A Physicist Looks at Mathematics

HOW DO PHYSICISTS view mathematics? Instead of answering this question by summarizing the writings of many physicists, we interviewed one physicist whose scientific feelings were judged to be representative. Since the summary which follows cannot represent his full and precise views, his name has been changed.

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