Wavelets for Elliptical Waveguide Problems

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Abstract: - New elliptic cylindrical wavelets are introduced, which exploit the relationship between analysing filters and Floquet's solution of Mathieu differential equations. It is shown that the transfer function of both multiresolution filters is related to the solution of a Mathieu equation of odd characteristic exponent. The number of notches of these analysing filters can be easily designed. Wavelets derived by this method have potential application in the fields of optics, microwaves and electromagnetism.

Key-Words: - Wavelets, Elliptic cylinder, Waveguide, Mathieu wavelets

1 Introduction

A family of linear second-order differential equation with periodic coefficients was introduced by Émile Mathieu when studying vibrations in an elliptic drum [1]. Mathieu's equation is related to the wave equation for the elliptic cylinder. This paper is concerned with the canonical form of the Mathieu Equation, i.e., given $a \in \mathbb{R}$, $q \in \mathbb{C}$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}w^2} + (a - 2q\cos(2w))y = 0. \tag{1}$$

The solution of Equation 1 is the elliptic cylindrical harmonic, known as Mathieu functions. Mathieu wavelets as well as Mathieu functions can be appealing in a lot of Physics issues [2] including vibrations in an elliptic drum, diffraction, amplitude distortion, the inverted pendulum, the radio frequency quadrupole, stability of a floating body, alternating gradient focusing, vibration in a medium with modulated density, and even when examining molecular dynamics of charged particles in electromagnetic traps [3, 4].

They have also long been applied on a broad scope of waveguide problems involving elliptical geometry [5, 6, 7, 8, 9, 10, 11, 12, 13], including: (i) analysis for weak guiding for step index elliptical core optical fibres, (ii) power transport of elliptical waveguides, (iii) evaluating radiated waves of elliptical horn antennas, (iv) elliptical annular microstrip antennas with arbitrary eccentricity, and (v) scattering by a coated strip. In general, the solutions of Equation 1 are not periodic. However, for a given q, periodic solutions exist for infinitely many special values (eigenvalues) of a. For many physical solutions y must be periodic of period π or 2π. It is also convenient to distinguish even and odd periodic solutions, which are termed Mathieu functions of first kind.

2 Preliminaries

One of four simpler types can be considered: Periodic solution (π or 2π) symmetry (even or odd). For $q \neq 0$, the only periodic solution y corresponding to any characteristic value $a = a_\nu(q)$ or $a = b_\nu(q)$ has the following notation:
Even periodic solution

$$\mathrm{ce}_\nu(w,q) = \sum_m A_{\nu,m} \cos mw \qquad \text{for } a = a_\nu(q), \tag{2a}$$

Odd periodic solution

$$\mathrm{se}_\nu(w,q) = \sum_m A_{\nu,m} \sin mw \qquad \text{for } a = b_\nu(q), \tag{2b}$$

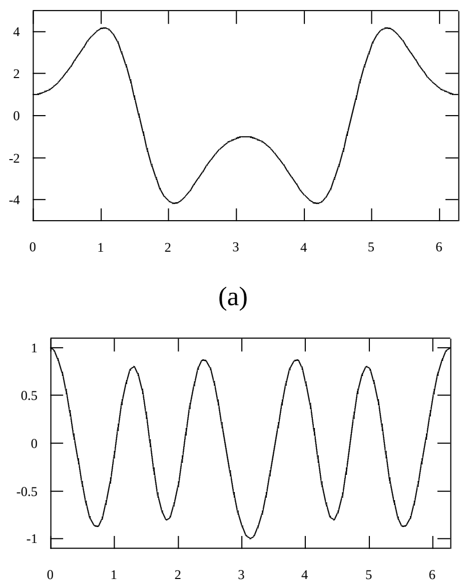
where the sums are taken over even (respectively odd) values of m if the period of y is π (respectively 2π). Given ν, we denote henceforth $A_{\nu,m}$ by A_m, for short. Elliptic cosine and elliptic sine functions are represented by ce and se, respectively. Interesting relationships are found when $q \to 0$, $\nu \neq 0$ [14]:

$$\lim_{q\to 0} \mathrm{ce}_\nu(w,q) = \cos(\nu w), \tag{3}$$

$$\lim_{q\to 0} \mathrm{se}_\nu(w,q) = \sin(\nu w). \tag{4}$$

As it happens with trigonometric functions, Mathieu functions hold orthogonality properties:

$$\int_0^{2\pi} \mathrm{ce}_\nu(w,q)\, \mathrm{ce}_\mu(w,q) \mathrm{d}w = 0,\ \nu \neq \mu, \tag{5}$$



(a)

(b)

Figure 1: Some plots of 2π-periodic 1st kind even Mathieu functions. Elliptic cosines shape for the following set of parameters: a) $\nu = 1$ and $q = 5$, b) $\nu = 5$ and $q = 5$.

$$\int_0^{2\pi} \mathrm{se}_\nu(w,q)\,\mathrm{se}_\mu(w,q)\mathrm{d}w = 0,\ \nu \neq \mu, \tag{6}$$

$$\int_0^{2\pi} \mathrm{ce}_\nu(w,q)\,\mathrm{se}_\mu(w,q)\mathrm{d}w = 0. \tag{7}$$

The second non-periodic solution [14] corresponding to $\mathrm{ce}_\nu(w,q)$ is the Mathieu function $\mathrm{Zce}_\nu(w,q)$. Similarly, The second non-periodic solution corresponding to $\mathrm{se}_\nu(w,q)$ is the Mathieu function $\mathrm{Zse}_\nu(w,q)$.

One of the most powerful results of Mathieu's functions is the Floquet's Theorem [15, 16]. It states that periodic solutions of Equation 1 for any pair (a,q) can be expressed in the form

$$y(w) = F_\nu(w) = e^{j\nu w}P(w) \qquad \text{or} \tag{8}$$

$$y(w) = F_\nu(-w) = e^{-j\nu w}P(-w), \tag{9}$$

where ν is a constant depending on a and q and $P(\cdot)$ is π-periodic in w. The constant ν is called the characteristic exponent. If ν is an integer, then $F_\nu(w)$ and $F_\nu(-w)$ are linear dependent solutions. Furthermore, $y(w+k\pi) = e^{j\nu k\pi}y(w)$ or $y(w+k\pi) = e^{-j\nu k\pi}y(w)$, for the solution $F_\nu(w)$ or $F_\nu(-w)$, respectively. We assume that the pair (a,q) is such that $|\cosh(j\nu\pi)| < 1$ so that the solution $y(w)$ is bounded on the real axis [17]. The general solution of Mathieu's equation ($q \in \mathbb{R}$, ν non-integer) has the form

$$y(w) = c_1 e^{j\nu w}P(w) + c_2 e^{-j\nu w}P(-w), \tag{10}$$

where c_1 and c_2 are arbitrary constants.

All bounded solutions —those of fractional as well as integral order— are described by an infinite series of harmonic oscillations whose amplitudes decrease with increasing frequency. In the wavelet framework we are basically concerned with even solutions of period 2π. In such cases there exist recurrence relations among the coefficients [14]:

$$(a-1-q)A_1 - qA_3 = 0, \tag{11}$$

$$(a-m^2)A_m - q(A_{m-2}+A_{m+2}) = 0, \tag{12}$$
$$m \geq 3,\ m \text{ odd}.$$

Different methods for computing Mathieu functions are available in the literature [18, 19].

Figure 1 shows two illustrative waveforms of elliptic cosines, whose shape strongly depends on the parameters ν and q.

2.1 Elliptic Cylindrical Coordinates

The v coordinates are asymptotic angles of confocal hyperbolic cylinders, which are symmetric with respect to the x axis. The u coordinates are confocal elliptic cylinders centred in the origin.

$$x = a\cosh(u)\cos(v) \tag{13}$$

$$y = a\sinh(u)\sin(v) \tag{14}$$

$$z = z, \tag{15}$$

where $u \in [0,\infty)$, $v \in [0, 2\pi)$ and $z \in (-\infty, +\infty)$.

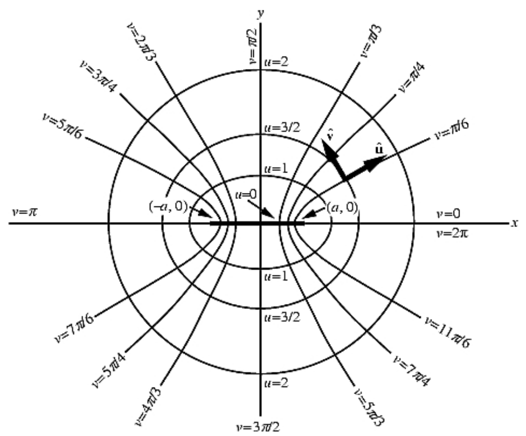


Figure 2: Elliptic cylindrical coordinates.

A further important property of Mathieu's functions is the orthogonality. If $a(\nu+2\pi,q)$ and $a(\nu+2s,q)$ are simple roots of $\cos\left(\frac{\pi}{2}\nu\right) - y\left(\frac{\pi}{2}\right)$, then [15]

$$\int_0^{\pi} F_{\nu+2p}(w)F_{\nu+2s}(-w)\mathrm{d}w = 0, \quad p \neq s. \tag{16}$$

In other words, $\langle F_{\nu+2p}(w), F_{\nu+2s}(-w)\rangle = 0$, $p \neq s$, where $\langle\cdot,\cdot\rangle$ denotes inner product.

3 Multiresolution Analysis Filters and Mathieu's Equation

Wavelet analysis has quickly developed over the past years [20] and there has been a explosion of wavelet applications including seismic geology, quantum physics, medicine, image processing (e.g., video data compression, reconstruction of high resolution images), fractals, computer graphics, linear system modelling, computer and human vision, denoising, filter banks, radar, wideband spreading, turbulence, statistics, volumetric visualisation, metallurgy, solution of partial differential equations, and so on. Essentially, the Wavelet Transform is a signal decomposition onto a set of basis functions, which is derived from a single prototype wavelet by scaling (dilatations and contractions) as well as translations (shifts). In the sequel, wavelets are denoted by $\psi(t)$ and scaling functions by $\phi(t)$, with corresponding spectra $\Psi(w)$ and $\Phi(w)$, respectively.

The equation $\phi(t) = \sqrt{2}\sum_{n\in\mathbb{Z}} h_n \phi(2t-n)$, which is known as the *dilation* or *refinement equation*, is the chief relation determining a Multiresolution Analysis (MRA) [21, 22].

3.1 Two Scale Relation of Scaling Function and Wavelet

Defining the spectrum of the smoothing filter $\{h_k\}$ by

$$H(w) \triangleq \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} h_k e^{-jwk}, \qquad (17)$$

the central equations (in the frequency domain) of a Multiresolution analysis are [22]:

$$\Phi(w) = H\left(\frac{w}{2}\right)\Phi\left(\frac{w}{2}\right), \qquad (18)$$

$$\Psi(w) = G\left(\frac{w}{2}\right)\Phi\left(\frac{w}{2}\right), \qquad (19)$$

where

$$G(w) \triangleq \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} g_k e^{-jwk} \qquad (20)$$

is the transfer function of the detail filter.

The orthogonality condition corresponds to [22, 20]:

$$H(0) = 1 \text{ and } H(\pi) = 0, \qquad (21)$$

$$|H(w)|^2 + |H(w+\pi)|^2 = 1, \qquad (22)$$

$$H(w) = -e^{-jw}G^*(w+\pi). \qquad (23)$$

3.2 Filters of a Mathieu MRA

The subtle liaison between Mathieu's theory and wavelets was found by observing that the classical relationship [20, 23]

$$\Psi(w) = e^{-jw/2}H^*\left(\frac{w}{2}-\pi\right)\Phi\left(\frac{w}{2}\right) \qquad (24)$$

presents a remarkable similarity to a Floquet's solution of a Mathieu's equation, since $H(w)$ is a periodic function.

As a first attempt, the relationship between the wavelet spectrum and the scaling function was put in the form:

$$\frac{\Psi(w)}{\Phi\left(\frac{w}{2}\right)} = e^{-jw/2}H^*\left(\frac{w}{2}-\pi\right). \qquad (25)$$

Here, on the second member, neither ν is an integer nor $H(\cdot)$ has a period π. By an appropriate scaling of this equation, we can rewrite it as

$$\frac{\Psi(4w)}{\Phi(2w)} = e^{-j2w}H^*(2w-\pi). \qquad (26)$$

It follows from Equation 17 that H is periodic. We recognise that the function $Y(w) \triangleq \Psi(4w)/\Phi(2w)$ has a nice interpretation in the wavelet framework. First, we recall that $\Psi(w) = G\left(\frac{w}{2}\right)\Phi\left(\frac{w}{2}\right)$ so that $\Psi(2w) = G(w)\Phi(w)$. Therefore the function related to Mathieu's equation is exactly $Y(w) = G(2w)$. Introducing a new variable z, which is defined according to $2z \triangleq 2w-\pi$, it follows that $-Y\left(z+\frac{\pi}{2}\right) = e^{-j2z}H^*(2z)$. The characteristic exponent can be adjusted to a particular value ν,

$$-e^{-j(\nu-2)z}Y\left(z+\frac{\pi}{2}\right) = e^{-j\nu z}H^*(2z). \qquad (27)$$

Defining now $P(-z) \triangleq H^*(2z) = \sum_{k\in\mathbb{Z}} c_{2k}e^{jz2k}$, where $c_{2k} \triangleq \frac{1}{\sqrt{2}}h_k^*$, we figure out that the right-side of the above equation represents a Floquet's solution of some Mathieu differential equation. The function $P(\cdot)$ is π-periodic verifying the initial condition $P(0) = \frac{1}{\sqrt{2}}\sum_k h_k = 1$, as expected. The filter coefficients are all assumed to be real. Therefore, there exist a set of parameters (a_G, q_G) such that the auxiliary function

$$y_\nu(z) \triangleq -e^{-j(\nu-2)z}Y_\nu\left(z+\frac{\pi}{2}\right) \qquad (28)$$

is a solution of the following Mathieu equation:

$$\frac{\mathrm{d}^2 y_\nu}{\mathrm{d}z^2} + (a_G - 2q_G\cos(2z))y_\nu = 0, \qquad (29)$$

subject to $y_\nu(0) = -Y(\pi/2) = -G(\pi) = -1$ and $\cos(\pi\nu) - y_\nu(\pi) = 0$, that is, $y_\nu(\pi) = (-1)^\nu$.

In order to investigate a suitable solution of Equation 29, boundary conditions are established for predetermined a, q. It turns out that when ν is zero or an integer, a belongs to the set of characteristic values $a_\nu(q)$. The even (2π-periodic) solution of such an equation is given by:

$$y_\nu(z) = -\frac{\mathrm{ce}_\nu(z,q)}{\mathrm{ce}_\nu(0,q)}. \qquad (30)$$

The $Y_\nu(w)$ function associated to $y_\nu(z)$ and related to the detail filter of a "Mathieu MRA" is thus:

$$Y_\nu(w) = G_\nu(2w) = e^{j(\nu-2)\left(w-\frac{\pi}{2}\right)}\frac{\mathrm{ce}_\nu\left(w-\frac{\pi}{2},q\right)}{\mathrm{ce}_\nu(0,q)}. \qquad (31)$$

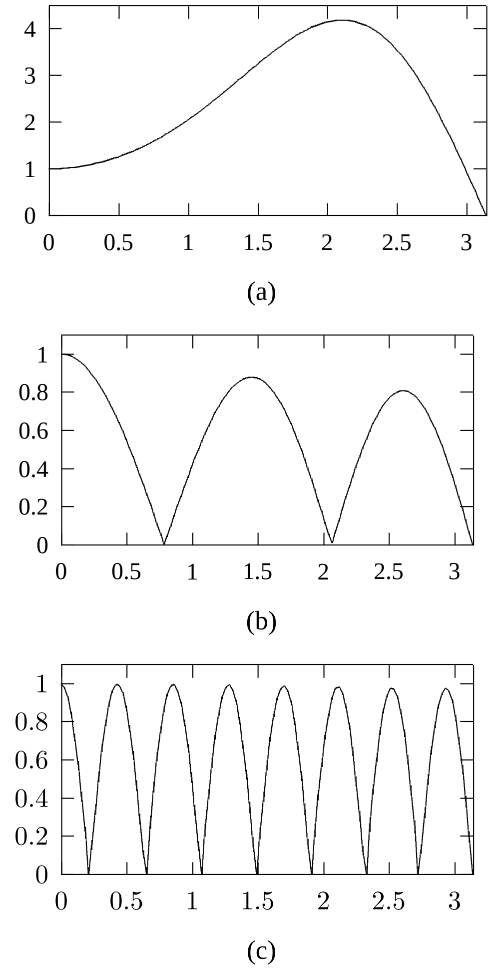


Figure 3: Magnitude of the transfer function for Mathieu multiresolution analysis: smoothing filter $|H_\nu(w)|$ for a few Mathieu parameters. (a) $\nu = 1$, $q = 5$, $a = 1.858$; (b) $\nu = 5$, $q = 5$, $a = 25.550$; (c) $\nu = 15$, $q = 5$, $a = 225.056$.

Finally, the transfer function of the detail filter of a Mathieu wavelet is

$$G_\nu(w) = e^{j(\nu-2)\left(\frac{w-\pi}{2}\right)} \frac{\mathrm{ce}_\nu\left(\frac{w-\pi}{2}, q\right)}{\mathrm{ce}_\nu(0,q)}. \qquad (32)$$

The characteristic exponent ν should be chosen so as to guarantee suitable initial conditions, i.e., $G_\nu(0) = 0$ and $G_\nu(\pi) = 1$, which are compatible with wavelet filter requirements [20, 23]. Therefore, ν must be odd. It is interesting to remark that the magnitude of the above transfer function corresponds exactly to the modulus of an elliptic sine [17]:

$$|G_\nu(w)| = \left|\mathrm{se}_\nu\left(\frac{w}{2}, -q\right) / \mathrm{ce}_\nu(0,q)\right|. \qquad (33)$$

The solution for the smoothing filter $H(\cdot)$ can be found out via QMF conditions [21], yielding:

$$H_\nu(w) = -e^{-j\nu\frac{w}{2}} \frac{\mathrm{ce}_\nu\left(\frac{w}{2}, q\right)}{\mathrm{ce}_\nu(0,q)}. \qquad (34)$$

In this case, we find $H_\nu(\pi) = 0$ and

$$|H_\nu(w)| = \left|\mathrm{ce}_\nu\left(\frac{w}{2}, q\right) / \mathrm{ce}_\nu(0,q)\right|. \qquad (35)$$

Given q, the even first-kind Mathieu function with characteristic exponent ν is given by

$$\mathrm{ce}_\nu(w,q) = \sum_{l=0}^{\infty} A_{2l+1} \cos(2l+1)w, \qquad (36)$$

in which $\mathrm{ce}_\nu(0,q) = \sum_{l=0}^{\infty} A_{2l+1}$. The G and H filter coefficients of a Mathieu MRA can be expressed in terms of the values $\{A_{2l+1}\}_{l\in\mathbb{Z}}$ of the Mathieu function as:

$$\frac{h_l^\nu}{\sqrt{2}} = -\frac{A_{|2l-\nu|}/2}{\mathrm{ce}_\nu(0,q)}, \qquad (37)$$

$$\frac{g_l^\nu}{\sqrt{2}} = (-1)^l \frac{A_{|2l+\nu-2|}/2}{\mathrm{ce}_\nu(0,q)}. \qquad (38)$$

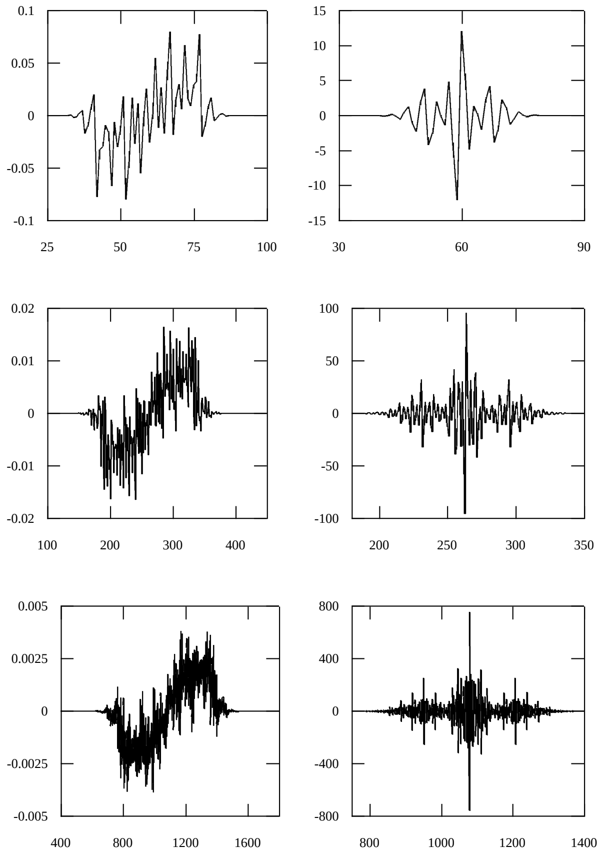
It is straightforward to show that $h_{-l}^\nu = h_{l+\nu}^\nu$, and $\forall l > 0$. The normalising conditions are $\frac{1}{\sqrt{2}} \sum_{k=-\infty}^{\infty} h_k^\nu = -1$ and $\sum_{k=-\infty}^{\infty} (-1)^k h_k^\nu = 0$.

Illustrative examples of filter transfer functions for a Mathieu MRA are shown in Figure 3, for $\nu = 1$ and 5, and a particular value of q (numerical solution obtained by 5-order Runge-Kutta method). The value of a is adjusted to an eigenvalue in each case, leading to a periodic solution. Such solutions present a number of ν zeroes in the interval $|w| < \pi$. We observe lowpass behaviour (for the filter H) and highpass behaviour (for the filter G), as expected. Mathieu wavelets can be derived from the lowpass reconstruction filter by an iterative procedure (the same approach as the one usually used for plotting Daubechies wavelets). Infinite Impulse Response filters should be applied since Mathieu wavelet has no compact support. However a Finite Impulse Response approximation can be generated by discarding negligible filter coefficients, say less than 10^{-10}.

In Figure 4, emerging pattern that progressively looks like the wavelet shape is shown. Waveforms were derived using the MATLAB wavelet toolbox. A fractal behaviour, which is common for some wavelets, can be noticed in these figures. As with many wavelets there is no nice analytical formula for describing Mathieu wavelets. Depending on the parameters a and q some waveforms (e.g. Fig. 4a) can present a somewhat unusual shape.

4 Conclusions

A new and wide family of elliptic cylindrical wavelets was introduced. It was shown that the transfer functions of the corresponding multiresolution filters are related to Mathieu equation solutions. The magnitude of the detail and smoothing filters corresponds to first-kind Mathieu functions with odd characteristic exponent. The number of zeroes of the highpass $|G(w)|$ and lowpass $|H(w)|$ filters within the interval $|w| < \pi$ can be appropriately



(a) (b)

Figure 4: FIR-Based Approximation of Mathieu Wavelets as the number of iteration increases (2, 4, and 6 iterations, respectively.) Filter coefficients holding $|h| < 10^{-10}$ were thrown away (20 retained coefficients per filter in both cases). (a) Mathieu Wavelet with $\nu = 1$ and $q = 5$ and (b) Mathieu Wavelet with $\nu = 5$ and $q = 5$.

designed by choosing the characteristic exponent. This seems to be the first connection found between Mathieu equations and wavelet theory. It opens new perspectives on linking wavelets and solutions of other differential equations (e.g. Associated Legendre functions, Coulomb wave function, Parabolic cylindrical functions etc.) Further generalisations such as the cases $q < 0$ or complex characteristic exponent can provide new interesting "waves". This new family of wavelets can particularly be an interesting tool for analysing optical fibres due to its symmetry [7, 10, 24]. Mathieu wavelets could as well be beneficial when examining molecular dynamics of charged particles in electromagnetic traps such as Paul trap or the mirror trap for neutral particles [3, 4].

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