von Mises Tapering: A Circular Data Windowing

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Introduction

Due to the cyclic (quasi-periodic) nature of many signals, the signal processing techniques for real variables may not be appropriate.

Circular measurements occur in many areas:

- biology (Chronobiology), epidemiology, medical (Circadian therapy)
- economy, geography
- geology, meteorology
- acoustic scatter
- GPS navigation
- oriented textures
- signal processing and over finite fields
- political analysis.

The **uniform distribution** $\phi \sim U(0, 2\pi)$ is:

$$f_{\Phi;1}(\phi) := \frac{1}{2\pi}\mathbb{I}_{[0,2\pi]}(\phi), \tag{1}$$

where $\mathbb{I}_A(.)$ is the indicator function of the interval $A \subset \mathbb{R}$.

The **normal circular distribution**, $\phi \sim VM(\phi_0, \beta)$ is:

$$f_{\Phi;2}(\phi) := \frac{1}{2\pi I_0(\beta)} e^{\beta \cos(\phi - \phi_0)}, \tag{2}$$

where $\beta \geq 0$ and $I_0(.)$ is the zero-order modified Bessel function of the first kind i.e.

$$I_0(z) := \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta} d\theta = \sum_{n=0}^{+\infty} \frac{(z/2)^{2n}}{n!^2}. \tag{3}$$

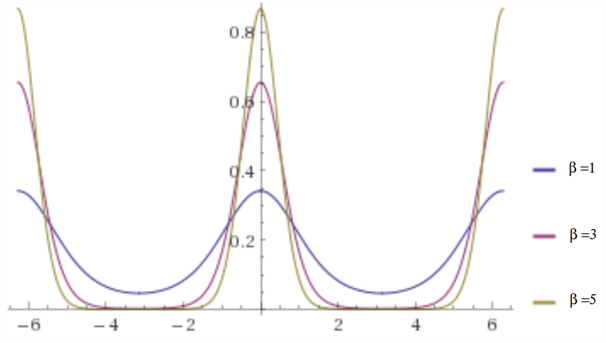

Figure 1: Periodic extension of the von Mises distribution with zero-mean for several parameter values: $\beta = 1,3,5$. Note that the support of the density is confined to $[-\pi, \pi]$.

In a standard notation,

$$f(x|\mu, \kappa) := \frac{e^{\kappa \cos(x-\mu)}}{2\pi I_0(\kappa)} \mathbb{I}_{[-\pi,\pi]}. \tag{4}$$

Two limiting behaviors can be observed:

$$\lim_{\kappa \to 0} VM(\mu, \kappa) \sim U(-\pi, \pi) \text{ and } \lim_{\kappa \to +\infty} VM(\mu, \kappa) \sim N(\mu, 1/\kappa). \tag{5}$$

it is known as the *circular normal distribution*.

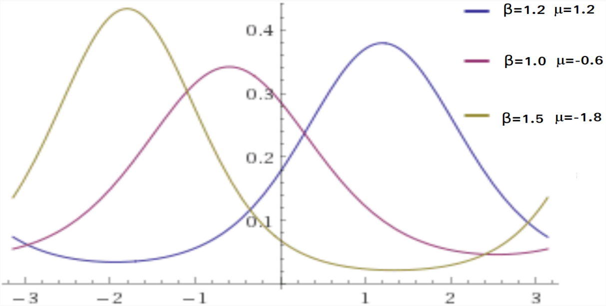

Figure 2: Circular behavior of the von Mises distribution plotted for different mean values (1.2, -0.6 and -1.8). The cyclical feature of the distribution within $[-\pi, \pi]$.

The support can be modified:

$$f_{X_1}(x) := \frac{e^{\beta \cos\left(\frac{2\pi}{N}x\right)}}{NI_0(\beta)}, \text{ circular in } 0 \leq x \leq N.$$

or

$$f_{X_2}(x) := \frac{e^{\beta \cos\left(\frac{\pi}{N}x\right)}}{NI_0(\beta)}, \text{ circular in } 0 \leq x \leq N.$$

Decaying pulses for constraining the signal support play a key role in: tapers, filtering, ISI interference, modulation, spectroscopy …

Standard Windows

continuous windows (**apodization function**):

rectangular, Bartlett, cosine-tip, Hamming, Hanning, Blackman, Lanczos, Kaiser, de la Vallé-Pousin, Poisson, (non-exhaustive list).

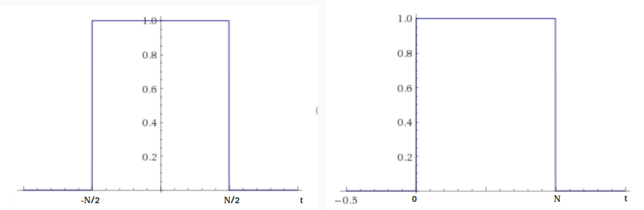

Figure 3: rectangular window, length N. a) continuous b) continuous causal.

$$w_{REC;1}(t) = \text{rect}\left(\frac{t}{N}\right),$$
$$w_{REC;2}(t) = \text{rect}\left(\frac{t-N/2}{N}\right).$$

In the continuous case, $w_1(t)$ has spectrum given by:

$$W(w) := \mathscr{F}[w(t)] = \int_{-\infty}^{+\infty} w(t)e^{-jwt}dt. \tag{6}$$

$W_{REC;1}(w) = N\text{Sa}\left(\frac{wN}{2}\right)$.

$W_{REC;2}(w) = N\text{Sa}\left(\frac{Nw}{2}\right)e^{-jNw/2}$.

Several windows can be encompassed by

$$w_{\alpha;1}(t) := \left\{\alpha + (1-\alpha)\cos\left(\frac{2\pi}{N}t\right)\right\}\text{rect}\left(\frac{t}{N}\right), \tag{7}$$

The **Hanning (Raised Cosine) window** corresponds to $\alpha = 0.5$, whereas the standard **Hamming window** corresponds to $\alpha = 0.54$.

In the case of a **cosine-tip continuous window** ($\alpha = 0$),

$$w_{\alpha=0;1} := \cos\left(\frac{2\pi}{N}t\right)\text{rect}\left(\frac{t}{N}\right), \tag{8}$$

and

$$W_{\alpha=0;1}(w) = \frac{N}{2}\text{Sa}\left(\frac{Nw}{2} - \pi\right) + \frac{N}{2}\text{Sa}\left(\frac{Nw}{2} + \pi\right). \tag{9}$$

The **Kaiser window** in continuous variable is defined by (non-causal window centered on the origin, and its corresponding causal version)

$$w_{KAI;1}(t) := \frac{I_0\left(\beta\sqrt{1-\left(\frac{t}{N/2}\right)^2}\right)}{I_0(\beta))}\operatorname{rect}\left(\frac{t}{N}\right), \tag{10a}$$

$$w_{KAI;2}(t) := \frac{I_0\left(\beta\sqrt{1-\left(\frac{t-N/2}{N/2}\right)^2}\right)}{I_0(\beta))}\operatorname{rect}\left(\frac{t-N/2}{N}\right). \tag{10b}$$

The corresponding spectrum is given by:

$$W_{KAI;1}(w) = \frac{N}{I_0(\beta)}\operatorname{Sa}\left(\sqrt{(\frac{Nw}{2})^2-\beta^2}\right). \tag{11}$$

Introducing a New Window: the Circular Normal Window

$$w(t) = K\frac{e^{\beta \cos\left(\frac{\pi}{N}t\right)}}{I_0(\beta)}\text{rect}\left(\frac{t}{N}\right). \tag{12}$$

The value of the constant K can be set so that, as in the other classic windows, $w(0) = 1$. Thus, for continuous case (noncausal and causal, respectively), one has

$$w_{CIR;1}(t) = \frac{e^{\beta \cos\left(\frac{\pi}{N}t\right)}}{e^{\beta}}\text{rect}\left(\frac{t}{N}\right), \tag{13a}$$

$$w_{CIR;2}(t) = \frac{e^{\beta \cos\left(\frac{\pi}{N}t\right)}}{e^{\beta}}\text{rect}\left(\frac{t - N/2}{N}\right). \tag{13b}$$

A straightforward comparison among different tapers in displayed:

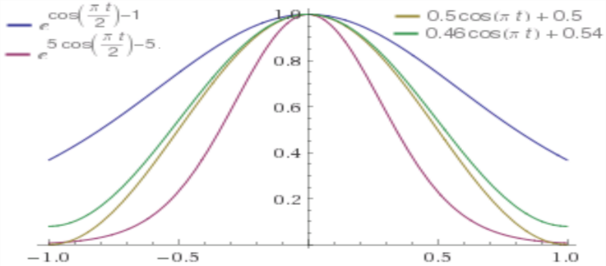

Figure 4: Windows shape: von Hann, Hamming, circular $\beta = 1,\ 5$.

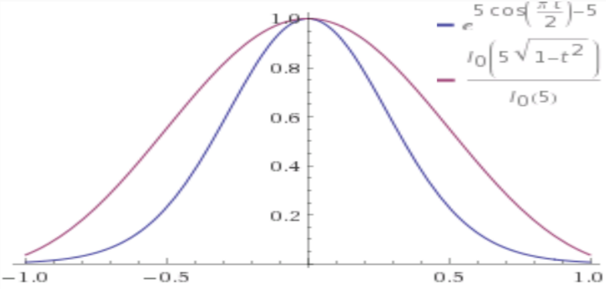

Figure 5: Windows shape: Kaiser vs normal circular windows, $\beta = 5$.

Spectrum calculation of the Normal Circular Window: the Continuous Case

Spectrum of the continuous window of the previous section

$$W_{CIR;1}(w) = \int_{-N/2}^{N/2} e^{\beta\left[\cos\left(\frac{\pi}{N}t\right)-1\right]} e^{-jwt} dt. \tag{14}$$

The interest function involved in defining the window is $\cos\left(\frac{\pi}{N}t\right)$, with period $2N$, sketched below in $[-N, N]$.

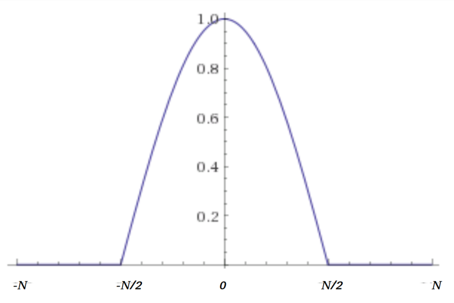

Figure 6: Normalized cosine exponent of the exponential function in von Mises window: the (entire) cosine $\cos(\pi t/N)$ is periodic in $[-N, N]$, but the support is confined within $[-N/2, N/2]$ due to the rectangular pulse.

MacLaurin's series development of $e^{\beta \cdot [\cos(\frac{\pi}{N}t)]}$ gives the following Fourier series:

$$e^{\beta[\cos(\frac{\pi}{N}t)]} = \sum_{n=-\infty}^{+\infty} I_{|n|}(\beta) \cos\left(\frac{n\pi}{N}t\right). \qquad (15)$$

Thus, one obtains:

$$W_{CIR;1}(w) = e^{-\beta} \sum_{n=-\infty}^{+\infty} I_{|n|}(\beta)\mathscr{F}\left(\cos\left(\frac{n\pi}{N}t\right)\operatorname{rect}\left(\frac{t}{N}\right)\right). \tag{16}$$

From the convolution theorem in the frequency:

$$W_{CIR;1}(w) =$$

$$= \frac{1}{2\pi}e^{-\beta} \sum_{n=-\infty}^{+\infty} I_{|n|}(\beta)\mathscr{F}\left(\cos\left(\frac{n\pi}{N}t\right)\right) * \mathscr{F}\left(\operatorname{rect}\left(\frac{t}{N}\right)\right). \tag{17}$$

or

$$W_{CIR;1}(w) = Ne^{-\beta} \sum_{n=-\infty}^{\infty} I_{|n|}(\beta)\left\{\operatorname{Sa}\left(\frac{Nw}{2} - \frac{n\pi}{2}\right)\right\}. \tag{18}$$

This expression is like a series of reconstitution (with coefficients c_n) of the type

$$\sum_{n=-\infty}^{+\infty} c_n \operatorname{Sa}\left(\frac{Nw}{2} - \frac{n\pi}{2}\right).$$

Let us apply the Shannon-Nyquist-Koteln'kov sampling theorem in the frequency domain, for time-limited signals `http://ict.open.ac.uk/classics`.

Since

$$F(w) = \frac{w_s t_m}{\pi} \sum_{n=-\infty}^{+\infty} F(n w_s) \mathrm{Sa}\left(w t_m - n t_m w_s\right). \tag{19}$$

The rate w_s must comply with the restriction $w_s \leq \pi / t_m$, and the choice made is $w_s = \pi / 2t_m$, so that the previous equation is

$$F(w) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} F(\frac{n\pi}{2t_m}) \mathrm{Sa}\left(w t_m - \frac{n\pi}{2}\right). \tag{20}$$

Now let us choose the duration t_m to be $t_m := N/2$ (Figure 6), which leads to

$$F(w) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} F(\frac{n\pi}{N}) \mathrm{Sa}\left(\frac{Nw}{2} - \frac{n\pi}{2}\right). \tag{21}$$

This is a variation of the cardinal Whittaker-Shannon series.

Observing the series described in Eq. (18):

$$F(w) = 2I_{\left|\frac{Nw}{\pi}\right|}(\beta), \tag{22}$$

and the spectrum is given by

$$W_{CIR;1}(w) = \frac{2NI_{\left|\frac{Nw}{\pi}\right|}(\beta)}{e^{\beta}}. \tag{23}$$

In the case of the causal window, $w_{CIR;1}(t)$, the application of the time-shift theorem provides the spectrum

$$W_{CIR;2}(w) = \frac{2NI_{\frac{N}{\pi}|w|}(\beta)}{e^{\beta}} e^{-jw\frac{N}{2}}. \tag{24}$$

It is worth remembering that the ν argument of the $I_{\nu}(z)$ function is a real number in this case.

Conclusions

The closeness to the normal distribution and the fact that they are associated with a shape linked to the maximum entropy for circular data suggests interesting properties to be explored in later investigations. Windowing circular data with von Mises circular window can possibly improve spectral evaluation in these cases. Discrete data windows of this kind is currently under investigation.