

The Hamming and Golay Number-Theoretic Transforms

A. J. A. Paschoal , R. M. Campello de Souza and H. M. de Oliveira

Abstract—New number-theoretic transforms are derived from known linear block codes over finite fields. In particular, two new such transforms are built from perfect codes, namely the *Hamming number-theoretic transform* and the *Golay number-theoretic transform*. A few properties of these new transforms are presented.

Keywords—Number-theoretic transforms (NTT), Finite fields, Linear block codes, Perfect codes, Perfect transforms.

I. PRELIMINARIES

Digital transforms are an interesting way to construct linear block codes for error correction [4], [5]. Fourier and Hartley codes were defined associated with the Fourier number-theoretic transform and the Hartley number-theoretic transform, respectively. Recently, a new family of multilevel codes – the Pascal codes [14] - has been proposed, which are constructed from the Pascal number-theoretic transform [13]. Also recently, Paschoal [14] proposed a reverse way to conceive new digital transforms from known linear block codes. Here we present the technique used to engender two new transforms over finite fields, namely the Hamming Number-Theoretic Transform (HamNT) and the Golay Number-Theoretic Transform (GolNT). From now on these transforms, based on Perfect Codes, are referred to as “Perfect Transforms”. Tietäväinen demonstrated in 1971 that the only existing perfect block codes over $GF(p)$ are [18] :

- Hamming codes $(N = \frac{p^m-1}{p-1}, k = \frac{p^m-1}{p-1} - m, d = 3)$.
- Binary Golay code $(N = 23, k = 12, d = 7)$.
- Ternary Golay code $(N = 11, k = 6, d = 5)$.

The perfect codes are also deeply linked to the lattices in Euclidean spaces [6]. Only the E_8 Gosset lattice (associated with the extended Hamming code) and the Leech lattice Λ_{24} (associated with the extended binary Golay code) are perfect sphere packings.

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II. NEW FINITE FIELD TRANSFORMS DERIVED FROM BLOCK CODES

In [4] Campello de Souza and coworkers used the fact that for every linear transform T , its *eigenvectors* v satisfy

$$[T - \lambda\mathbb{I}]v = 0, \quad (1)$$

where λ denotes the eigenvalue associated with v , to show that it is possible to get an associated parity-check matrix for a block code simply by row reducing the matrix $[T - \lambda\mathbb{I}]$. In this paper we go on a somewhat opposite direction, i.e., starting from a linear block code, it is possible to derive a new linear transformation (associated with the generating block code) by completing the parity-check matrix, adding k rows, so as to form a square matrix H_e and adding it to the matrix $\lambda\mathbb{I}$, i.e.,

$$T = H_e + \lambda\mathbb{I}. \quad (2)$$

Indeed, λ must be chosen so as to guarantee that $\det T \neq 0$. Such transform receives the name of the linear block code whose parity matrix was used to derive it.

The number of vectors on the whole space of all the N -tuples over $GF(p)$ is p^N (with no redundancy). The set of all eigenvectors of a transform of length N defined over a finite field always engender a subspace denoted by \mathbb{V} , with dimension $k < N$ with $GF(p)$ -valued components.

Definition 1: (perfect transform). A transform T of length N over $GF(p)$ is said to be perfect *if and only if* there is an integer t such that the eigenvectors subspace \mathbb{V} of T has dimension meeting the sphere-packing bound

$$\dim \mathbb{V} = k = N - \log_p \left[\sum_{i=0}^t (p-1)^i \binom{N}{i} \right]. \quad (3)$$

The eigenvector subspace of T is a perfect block code [18] when T is a perfect transform.

III. THE HAMMING NUMBER-THEORETIC TRANSFORM

In this section, we illustrate the aforementioned technique splitting it in two ways:

- a) Fulfilling the Hamming parity-check matrix by adding null rows or using linear combinations of its own rows.
- b) Cyclic rotating the parity polynomial, $h(x)$.

A. The Standard Hamming Number-Theoretic Transform

Consider the binary Hamming code $\mathcal{H}(7, 4, 3)$, with parity-check matrix (reduced row echelon form)

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

In this case, in order to construct the Hamming Number-Theoretic Transform (HamNT) matrix, $T_{HamNT}^{(\lambda)}$, we need to add four rows to this parity matrix to get the matrix H_e . Such additional rows are merely null rows. This new matrix, named H_e , is added to $\lambda \mathbb{I}_N$, where $\lambda = 1$ is the eigenvalue over $GF(2)$ and $\mathbb{I}_N = \mathbb{I}_7$ is the identity matrix of dimension 7. Note that, in fact, the eigenvalues are unknown and we should test all the possible candidates eliminating all those that result in a singular transformation matrix. The different eigenvalues may produce different transformation matrices. From the binary Hamming code, we get the 7×7 binary HamNT whose transformation matrix is

$$T_{HamNT}^{(1)} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvector matrix is given by

$$\text{eig } T_{HAM}^{(1)} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} = G,$$

which is the generator matrix of the Hamming code $\mathcal{H}(7, 4, 3)$. Note that $7 - \log_2(1 + 7) = 4 = \dim \mathbb{V}$. Figure 1 shows the implementation of the binary Hamming Transform 7×7 .

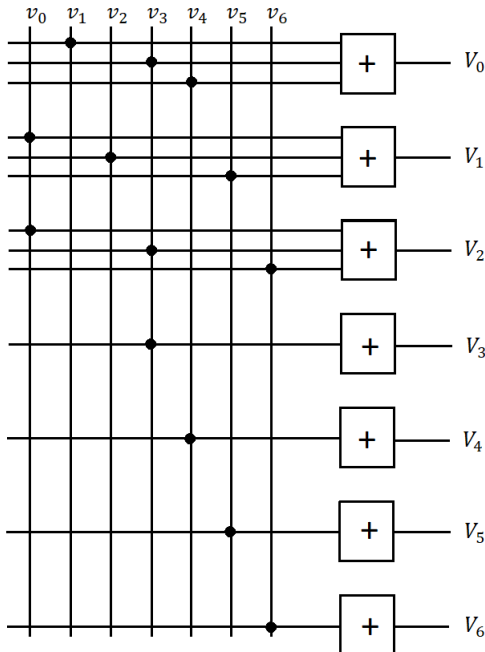


Fig. 1

BLOCK DIAGRAM FOR THE BINARY HAMMING TRANSFORM OF LENGTH 7.

Definition 2: The Hamming Number-Theoretic Transform, with length $N = (p^m - 1)/(p - 1)$, of the sequence $\mathbf{v} =$

$(v_0, v_1, \dots, v_{N-1})^T$, $v_i \in GF(p)$, is the sequence $\mathbf{V} = (V_0, V_1, \dots, V_{N-1})^T$, $V_k \in GF(p)$, given by

$$\mathbf{V} = T_{HAM}^{(\lambda)} \cdot \mathbf{v}, \quad (4)$$

where $T_{HAM}^{(\lambda)}$ is the Hamming transformation matrix over $GF(p)$, parametrized by eigenvalue λ .

The only natural property is that this transformation is linear, since the distributive property of multiplication relative to addition holds by matrices. Furthermore, all the codewords of the classic Hamming code are invariant under the HamNT.

Example 1: (nonbinary Hamming transform). Consider the ternary Hamming code $\mathcal{H}(13, 10, 3)$ whose parity-check matrix is

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}.$$

The transform matrix of the ternary Hamming transform, with eigenvalue $\lambda = 1$, is given by

$$T_{HAM}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This kind of construction makes it harder to find the properties of such a transform. In what follows we adopt a polynomial approach.

B. The Cyclic Hamming Transform

Another way to represent the Hamming transform over $GF(p)$ introduced in the previous subsection is considering the parity-check matrix H as expressed in the form

$$H = \begin{bmatrix} h(x) \\ xh(x) \\ x^2h(x) \\ \vdots \\ x^{n-k-1}h(x) \end{bmatrix},$$

where $h(x)$ is the parity polynomial of the cyclic Hamming code over $GF(p)$ [12]. The k rows necessary to construct the matrix H_e are generated by cyclic shifting the parity polynomial, $h(x)$.

Example 2: Consider the binary Hamming cyclic code $\mathcal{H}(7, 4, 3)$ with parity polynomial given by $h(x) = x^4 + x^2 + x + 1$. The parity-check matrix H is given by

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

The transformation matrix, in this case, is

$$T_{HAM}^{(\lambda)} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} + \lambda \mathbb{I}_7.$$

Taking $\lambda = 1$ results in the cyclic form of the binary Hamming transformation matrix (the inverse matrix is also circulant)

$$\tilde{T}_{HAM}^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The transform matrices $T_{HAM}^{(1)}$ and $\tilde{T}_{HAM}^{(1)}$, illustrated in Fig.2, are different representations for the Hamming transform over $GF(2)$. Note that, by considering the rows of $\lambda \mathbb{I}_N$ as cyclic shifts of the vector $[\lambda 00 \cdots 0]$, in polynomial notation, we can write

$$\lambda \mathbb{I}_N = \begin{bmatrix} \lambda(x) \\ x\lambda(x) \\ x^2\lambda(x) \\ \vdots \\ x^{N-1}\lambda \end{bmatrix}.$$

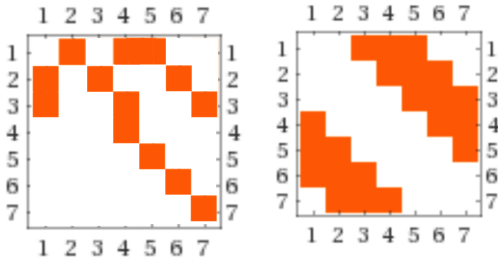


Fig. 2

ILLUSTRATION OF HAMMING TRANSFORMS: $T_{HAM}^{(1)}$, $\tilde{T}_{HAM}^{(1)}$.

Definition 3: The Cyclic Hamming Number-Theoretic Transform (CHamNT), with length $N = (p^m - 1)/(p - 1)$, of the sequence $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{N-1})^T$, $v_i \in GF(p)$, is the sequence $\mathbf{V} = (\mathbf{V}_0, \mathbf{V}_1, \dots, \mathbf{V}_{N-1})^T$, $V_k \in GF(p)$, such that $\mathbf{V} = \tilde{T}_{HAM}^{(\lambda)} \cdot \mathbf{v}$, where

$$T_{HAM}^{(\lambda)}(x) = \begin{bmatrix} [h(x) + \lambda(x)] \\ x[h(x) + \lambda(x)] \\ x^2[h(x) + \lambda(x)] \\ \vdots \\ x^{N-1}[h(x) + \lambda(x)] \end{bmatrix}.$$

Note that this definition paves the way for an algebraic treatment of the properties of the CHamNT.

Proposition 1: The matrix $T_{HAM}^{(1)}$ is a circulant matrix.

The procedure to generate H_e by means of cyclic shifts of the parity polynomial, $h(x)$, results in a circulant matrix. This aspect is not modified by adding $\lambda \mathbb{I}_N$ to this matrix.

C. Some of Properties of the Cyclic Hamming Number-Theoretic Transform

i) **Linearity**

ii) **Time shift**

Consider the sequence $\hat{\mathbf{v}} = (\hat{v}_0, \dots, \hat{v}_{N-1})$ where $\hat{v}_i = v_{i-m}$. Then, $\hat{\mathbf{v}} \leftrightarrow \hat{\mathbf{V}}$, where

$$\hat{\mathbf{V}} = x^m \mathbf{V}. \quad (5)$$

Demonstration: Note that we are considering, without loss of generality, a cyclic shift of m positions to the right. Therefore, we have $\hat{\mathbf{v}} = x^m \mathbf{v} \pmod{x^N - 1}$. Then,

$$\begin{aligned} \hat{\mathbf{V}} &= \begin{bmatrix} [h(x) + \lambda(x)] \\ x[h(x) + \lambda(x)] \\ x^2[h(x) + \lambda(x)] \\ \vdots \\ x^{N-1}[h(x) + \lambda(x)] \end{bmatrix} \begin{bmatrix} x^m v_0 \\ x^m v_1 \\ x^m v_2 \\ \vdots \\ x^m v_{N-1} \end{bmatrix} \\ &= x^m \begin{bmatrix} [h(x) + \lambda(x)] \\ x[h(x) + \lambda(x)] \\ x^2[h(x) + \lambda(x)] \\ \vdots \\ x^{N-1}[h(x) + \lambda(x)] \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \\ &= x^m \mathbf{V}. \end{aligned}$$

iii) **Frequency shift**

Consider the sequence $\hat{\mathbf{V}} = (\hat{V}_0, \dots, \hat{V}_{N-1})$ where $\hat{V}_k = V_{k-l}$. Then, $\hat{\mathbf{v}} = x^l \mathbf{v}$. The demonstration is similar to the previous one.

iv) **Constant sequence transform**

The transform of a sequence $\mathbf{v} = (r, r, \dots, r)$ is the sequence $V_k = r \cdot \text{weight}(h(x))$, $\forall k$.

This property means that the Hamming transform of a constant vector is also a constant vector. This contrasts sharply with the discrete Fourier transform, where there is a trade-off between the two domains and compressing into one domain implies expanding into the other.

v) **Impulse sequence transform**

The transform of the sequence $\delta = (1, 0, \dots, 0)$, corresponds to the first column of the matrix $T_H^{(\lambda)}$, i.e., the coefficients of $x[h(x) + \lambda(x)]$.

IV. THE GOLAY NUMBER-THEORETIC TRANSFORM

There exist two Golay codes, the binary $\mathcal{G}(23, 12, 7)$ ($h(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^5 + x^2 + 1$) and the ternary $\mathcal{G}(11, 6, 5)$ ($h(x) = x^6 + 2x^5 + 2x^4 + 2x^3 + x^2 + 1$).

Like in the previous section, to define the Golay number theoretic transform, we adopt a polynomial approach.

A. The Binary Golay Transform

The construction of the transformation matrix of the Cyclic Golay Transform follows the same steps used in the construction of the Cyclic Hamming Transform. Note that $2^{12} [1 + 23 + 253 + 1771] = 2^{23}$ and that $23 - \log_2(1 + 23 + 253 + 1771) = 12 = \dim \mathbb{V}$.

The Golay binary $\mathcal{G}(23, 12, 7)$ code has parity polynomial $h(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^5 + x^2 + 1$, so that

$$T_{CGolNT}^{(\lambda)}(x) = \begin{pmatrix} 011110010010100000000000 \\ 001111001001010000000000 \\ 000111100100101000000000 \\ 000011110010010100000000 \\ 000001111001001010000000 \\ 000000111100100101000000 \\ 000000011110010010100000 \\ 000000001111001001010000 \\ 000000000111100100101000 \\ 000000000011110010010100 \\ 000000000001111001001010 \\ 000000000000111100100101 \\ 100000000000011110010010 \\ 010000000000001111001001 \\ 101000000000000111100100 \\ 010100000000000011110010 \\ 001010000000000001111001 \\ 100101000000000000111100 \\ 010010100000000000011110 \\ 001001010000000000001111 \\ 100100101000000000000111 \\ 110010010100000000000011 \\ 111001001010000000000001 \\ 111100100101000000000000 \end{pmatrix},$$

the determinant of which is equals to one.

B. The Ternary Golay Transform

The Golay ternary parity polynomial is $h(x) = x^6 + 2x^5 + 2x^4 + 2x^3 + x^2 + 1$. Considering $\lambda = 1$ as eigenvalue, we have the following circulant form for the Golay transform matrix:

$$T_{CGolNT}^{(1)} = \begin{pmatrix} 2 & 2 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 \\ 2 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

The ternary eigenvector space of the Golay transform consists of $3^6 = 729$ vectors [16]. According to Sagemath[®], this transformation matrix has multiplicative order 242, characteristic polynomial $(2 + x)^6(1 + x + x^2 + x^3 + 2x^4 + x^5)$ and $\lambda = 1$ is an eigenvalue with algebraic multiplicity 6.

In the next example, we consider the construction of the ternary Golay transform starting from the systematic form of the Golay code parity-check matrix.

Example 3: Consider the ternary Golay code $\mathcal{G}(11, 6, 5)$ whose parity-check matrix is

$$H = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We inflate this matrix to a square matrix by adding null rows. To this matrix we add $\lambda \mathbb{I}_{11}$. Assuming $\lambda = 1$, it leads to

$$T_{GolNT}^{(1)} = \begin{pmatrix} 2 & 1 & 1 & 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This transform has the characteristic polynomial $p(x) = 1 + 2x^3 + 2x^4 + x^5 + 2x^6 + 2x^7 + x^8 + x^9 + 2x^{10} + x^{11}$, eigenvalue $\lambda = 1$ and determinant equals to 2. In this transform, the eigenvector subspace associated with $\lambda = 1$ has dimension $\dim \mathbb{V} = 11 - \log_3(1 + 22 + 220) = 6$. Note that $T_{CGolNT}^{(1)}$ and $T_{GolNT}^{(1)}$ are different representations for the Golay transform of length 11 over $GF(3)$, the first one being in the form of a circulating matrix.

C. A few properties of the cyclic Golay Number-Theoretic Transform

i) Linearity

ii) Time shift

Consider the sequence $\hat{\mathbf{v}} = (\hat{v}_0, \dots, \hat{v}_{N-1})$ where $\hat{v}_i = v_{i-m}$. Then, $\hat{\mathbf{v}} \leftrightarrow \hat{\mathbf{V}}$, where

$$\hat{\mathbf{V}} = x^m \mathbf{V}.$$

iii) Frequency shift

Consider the sequence $\hat{\mathbf{V}} = (\hat{V}_0, \dots, \hat{V}_{N-1})$ where $\hat{V}_k = V_{k-l}$. Then, $\hat{\mathbf{v}} = x^l \mathbf{v}$.

iv) Constant sequence transform

The transform of the sequence $\mathbf{v} = (r, r, \dots, r)$ is the sequence $V_k = r \cdot \text{weight}(h(x))$, $\forall k$.

v) Impulse sequence transform

The transform of the sequence $\delta = (1, 0, \dots, 0)$, corresponds to the first column of the matrix $T_{CGolNT}^{(\lambda)}$, i.e., the coefficients of the polynomial $x[h(x) + \lambda(x)]$.

These properties hold for both the binary and ternary Golay transform, simply using the corresponding parity polynomial.

V. GOLAY EXTENDED TRANSFORM

The extended versions of the Hamming or Golay codes, which are *selfdual* codes [6], can be used to construct new transforms. For the ternary Golay codes, the extended Golay code has parameters $\mathcal{G}(N = 12, k = 6, d = 6)$ over $GF(3)$. One possible approach over this field is to take into account that $2 \equiv -1$ and the transform matrix can be viewed only as computing additions (multiplication free transform). It should be interesting to compare it with Hadamard transforms [9]. The extended Golay parity-check matrix is given by

$$H = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Inflating this matrix to a square matrix by adding the following (arbitrary) linear combinations: $l_1 + l_2$, $l_1 + l_3$, $l_1 + l_4$, $l_1 +$

l_5 , $l_1 + l_6$ e $l_2 + l_3$, then adding $\lambda \mathbb{I}_{12}$, and assuming $\lambda = 1$, results in the 12×12 transform matrix

$$T_{EG}^{(1)} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

i.e., the matrix of the ternary extended Golay transform ($\det T_{EG}^{(1)} = 2$). This new transform requires only additions/subtractions in order to be computed. Their symmetries properties inherited from the selfdual structure of the space of eigenvectors can be attractive. The inverse transform matrix T_{EG}^{-1} is

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 1 & 1 & 0 & -1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & -1 & -1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & -1 & 0 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & -1 & -1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & -1 & 0 & 0 & -1 \end{bmatrix},$$

Finally, it is worth mentioning that since the eigenvalue is unity for the direct transform, all eigenvectors of the inverse transform are the same for the direct transform.

VI. FINAL REMARKS

The introduction of the Hamming (HamNT) and Golay (GolNT) number-theoretic transforms represents an important application of the theory introduced in [4], [5] and outlines the existence of some kind of *isomorphism* between linear block codes and finite field transforms. All the codewords of the classic Hamming code are invariant under HamNT (the same happens for the Golay code and the GolNT). A probable link between the Golay transform and Mathieu's simple group deserves to be investigated [6], [17]. These transforms can be used in Image Processing [3]. Even if it is an introductory work of presenting new ideas, without investigating potential applications, the strength of geometric configurations linked to perfect codes [2], [10] seems to give an inkle of the potential of these new transforms. This also opens up interesting perspectives in designing fast algorithms for the computation of these transformations [1], [11].

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APPENDIX

A general formulation of a discrete transform from systematic block codes: In the general case, starting from a block code with parity-check matrix

$$H = \left[\begin{array}{c|c} -P_{(N-k) \times k}^T & \mathbb{I}_{(N-k) \times (N-k)} \end{array} \right]$$

we arrive at a square transform matrix, $N \times N$, invertible, given by

$$T_{\text{block code}} = \left[\begin{array}{c|c} \lambda \mathbb{I}_{(N-k) \times k} - P_{(N-k) \times k}^T & \mathbb{I}_{(N-k) \times (N-k)} \\ \hline \mathbb{O}_{k \times (N-k)} & \lambda \mathbb{I}_{k \times k} \end{array} \right].$$