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# On generalized two-dimensional cross constellations and the opportunistic secondary channel*

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**Abstract**

*An optimum bits-to-symbol mapping for square constellations is introduced which allows a simple detection in narrow-sense quadrature amplitude modulation (QAM) systems. Then, a family of two-dimensional generalized cross constellations is presented as well as upper and lower bounds on the symbol error probability over an ideal band-limited channel which generalize those previously known for conventional QAM. The application of this scheme to the opportunistic secondary channel is analysed and it is shown how fractional rates (in bits per 2-dimensional signal) can be supported on 2D generalized QAM systems. These signaling schemes are compared with multidimensional generalized constellations recently proposed by Forney and Wei.*

**Key words :** Communication theory, Signal set, Quadrature modulation, Error probability, Transmission channel.

## SUR LES CONSTELLATIONS EN CROIX GÉNÉRALISÉES À DEUX DIMENSIONS ET LE CANAL SECONDAIRE OPPORTUNISTE

**Résumé**

*On introduit un étiquetage binaire optimal des symboles d'une constellation carrée. Il permet une détection simple pour les systèmes de modulation d'amplitude en quadrature (MAQ) au sens étroit. Ensuite, une famille de constellations en croix généralisées, à deux dimensions, est présentée ainsi que des bornes supérieure et inférieure de leur probabilité d'erreur par symbole sur un canal idéal à bande limitée, qui généralisent les bornes déjà connues pour la MAQ classique. L'application de ce schéma à un canal secondaire opportuniste est analysée et l'on montre comment des débits fractionnaires (en bits par signal à deux dimensions) peuvent être obtenus sur des systèmes MAQ généralisés. Ces schémas de communication sont comparés aux constellations multidimensionnelles généralisées récemment introduites par Forney et Wei.*

**Mots clés :** Théorie communication, Jeu signal, Modulation quadrature, Probabilité erreur, Canal transmission.

**Contents**



## I. INTRODUCTION

Digital schemes with coherent amplitude and phase modulation have long been used in data communication

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systems [1]. A clear example of this use is the broad commercial success of quadrature amplitude modulation (QAM) in both modems on telephone lines and digital microwave radio links. Over the additive white Gaussian noise channel (AWGN), Shannon's capacity theorem asserts that the maximum achievable rate (in bits per dimension) with virtually error-free communication solely depends on the signal-to-noise ratio (SNR). However, it seems more interesting to consider a normalized rate in bits per two dimensional signal (bits/2D) rather than in bits per dimension. For a band-limited channel with a bandwidth of $B$ hertz, the normalized rate exactly equals the spectral efficiency in (bit/s)/Hz of a system admitting a signaling rate of $2B$ dim/s [2].

The average noise power in the channel is $N = N_0B$, $N_0$ being the noise one-sided power spectral density. Assuming that the average received signal power is constrained to be equal to $S$, then the capacity $C$ of the Gaussian channel is given by :

(1) $C = \log_2(1 + S/N)$ bits/2D or (bit/s)/Hz.

A more conservative estimate of the possible rate is the cutoff rate $R_0$ which for this channel is given by [3] :

(2) $R_0 = \log_2\left(1 + \frac{S}{2N}\right)$ bit/2D or (bits/s)/Hz.

Let us consider the telephone channel as an example. Of course, other impairments besides the additive noise are present on the voiceband channel, but we assume that disturbances like nonlinear distortion, phase jitter, frequency offset, etc, are controlled on private special conditioned lines so that the additive Gaussian noise becomes the main impairment. For a signal-to-noise ratio of about 28 dB, the following limits are found : $R_0 \approx$ 8 bits/2D and $C \approx 9$ bits/2D. The ceiling 19.2 bits/s proposed in [4] corresponds exactly to the cutoff rate above when $B$ = 2,400 Hz. At present, there are commercial modems (with coding) working at 7 bits/2D or 7 (bits/s)/Hz, a rate fairly close to the cutoff rate. The rate increase from near the cutoff rate towards the capacity can be expected to be very hard. The usefulness of fractional rates in such a region becomes obvious. Still, transmitting a non-integral number of bits/2D is becoming common with the advent of coded multidimensional constellations, after Wei's paper [5]. In coded systems, a precise coding gain estimate involves its comparison with an uncoded system of same spectral efficiency, without sacrificing the rate nor requiring more bandwidth [6]. Section II excepted, this paper is mainly concerned with transmitting at fractional rates and with uncoded modulation schemes able to support these rates. The aim of this work is not to find good (dense) constellations but rather to provide a better understanding about the error probability calculation and the evaluation of the rate supported by constellations.

Much of the theory of the $N$-dimensional constellations appeared in [7]. This paper is very comprehensive but somewhat hard to read. We restrict ourselves here to the 2-dimensional case, which enables us to provide a (hopefully) easier introduction to the subject and to furthermore propose some refinements.

The paper is organized as follows. First, square constellations are considered in section II. An optimal bits-to-symbol assignment (2D Gray mapping) is presented which minimizes the bit error probability on AWGN channels and enables a simple implementation of uncoded QAM systems, particularly for large signal sets. Digital signaling methods using 2D generalized cross constellations (especially those with quadrilateral symmetry) are then discussed in section III. Both the average and peak power requirements of such constellations are evaluated. Bounds on the symbol error probability in the presence of AWGN are given. The *opportunistic secondary channel* is discussed in section IV as a means to employ generalized 2D QAM for transmitting a non-integral number of bits per two-dimension signal. The results thus obtained are compared with those of multidimensional generalized constellations [7]. The paper ends, in section V, with concluding remarks.

## II. A BITS-TO-SYMBOL ASSIGNMENT AND A SIMPLE DETECTION ALGORITHM FOR SQUARE QAM CONSTELLATIONS

In this section, we present a particular bits-to-symbol assignment for an $M$-point square constellation, $M$ a power of 4, to be referred to as 2D *Gray mapping*, which is optimum in the sense that it results in a minimum bit error probability over the Gaussian channel and which moreover enables a simple detection.

It is well known that no bits-to-symbol assignment exists such that a one-to-one correspondence between the Hamming and squared Euclidean distances (abbreviated as HD and ED, respectively) results, except for 2- or 4-point constellations. Nevertheless, in the case of some particular constellations we can design a mapping where the minimum ED increases as the HD increases (the key word here is *minimum*). Figure 1 below exhibits such a mapping.

A $(4M)$-point constellation is generated iteratively from an $M$-point one by substituting the points of the latter for a 4-point square constellation, to be referred to simply as a square. The initial constellation is the square whose points are labelled according to figure 1a. The process of iteratively labelling the points of the $(4M)$-point constellation is described as follows.

The first $\log_2 M$ bits which label all the points of a given square are identical to the label of the point of the $M$-constellation it replaces. Now, the last two bits of the labels are assigned to the 4 points of each square according to one of the 8 equivalent Gray codings of a square, one of which is illustrated in figure 1a. The choice of the particular Gray coding used for a given square is such that each point bears the same last-2-bit label as its closest neighbour belonging to another square. Therefore, the Gray patterns chosen to determine the last two bits are symmetric with respect to

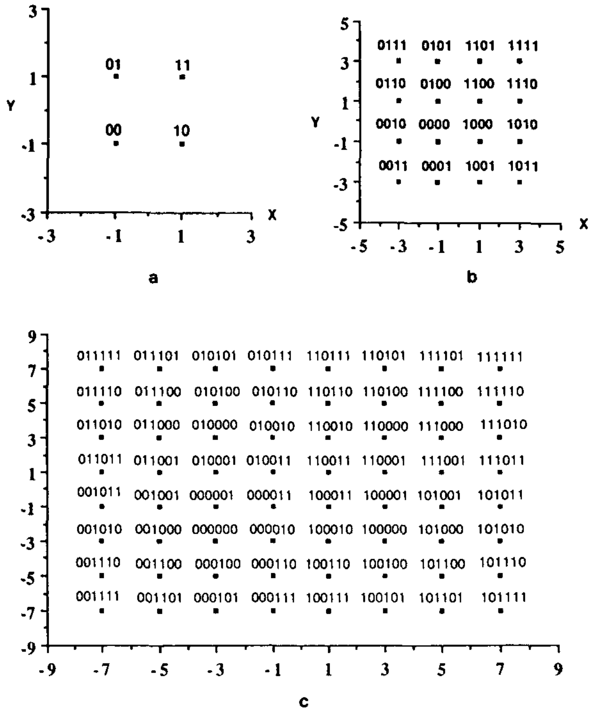


FIG. 1. — This figure illustrates the iterative construction and bits-to-symbol assignment of a (4*M*)-point square constellation from an *M*-point one. Figure 1a is the starting 4-point constellation with Gray coding bits-to-symbol assignment. The 16-point constellation of figure 1b results from replacing each point of the previous one by a 4-point constellation : the bits-to-symbol assignment used is described in the text. The next step of this process i.e., the 64-point constellation, is illustrated in figure 1c.

*Cette figure illustre la construction itérative et l'application bits à symbole d'une constellation carrée à 4M points à partir d'une constellation à M points. La figure 1a représente la constellation de départ, à 4 points, avec une application bits à symbole selon le code de Gray. La constellation à 16 points de la figure 1b est obtenue en remplaçant chacun des points de la précédente par une constellation à 4 points ; l'application bits à symbole utilisée est décrite dans le texte. L'étape suivante de ce processus, la constellation à 64 points, est illustrée sur la figure 1c.*

the boundaries of the squares. Clearly, 4 different Gray patterns are used, depending on the labelling of the initial 4-point constellation : the initial pattern, the two ones which result from it by symmetry with respect to the $x$- and the $y$-axes, and the one resulting from the product of the two symmetries i.e., the original pattern rotated by $\pi$.

If the Gray number $n_G$ is defined as the average number of erroneous bits per erroneous symbol [8], assuming only errors in favour of the closest points, it is apparent that $n_G = 1$ for this assignment (compare this with CCITT recommendation V32 [9]).

Let $\boldsymbol{b}$ and $\boldsymbol{b}'$ be two binary words (labels) assigned to the signal points $\boldsymbol{P}$ and $\boldsymbol{P}'$, respectively. Let $d_H$ (resp. $d_E$) denote the Hamming (resp. Euclidean) distance. Then, the two measures HD and ED are related according to the following inequalities :

$$(3)\quad \text{If}\quad \begin{array}{l} d_H(\boldsymbol{b},\boldsymbol{b}') = 1 \\ d_H(\boldsymbol{b},\boldsymbol{b}') = 2 \\ d_H(\boldsymbol{b},\boldsymbol{b}') = 3 \\ d_H(\boldsymbol{b},\boldsymbol{b}') = 4 \\ \text{etc} \end{array} \quad \text{then} \quad \begin{array}{l} d_E(\boldsymbol{P},\boldsymbol{P}') \geq \sqrt{1}d_0 \\ d_E(\boldsymbol{P},\boldsymbol{P}') \geq \sqrt{2}d_0 \\ d_E(\boldsymbol{P},\boldsymbol{P}') \geq \sqrt{5}d_0 \\ d_E(\boldsymbol{P},\boldsymbol{P}') \geq \sqrt{8}d_0 \\ \text{etc} \end{array}$$

where $d_0$ is the minimum Euclidean distance between two constellation points.

It can readily be seen that the most likely error patterns result in less faulty detected bits than any other bits-to-symbol assignment, and consequently that the bit error probability is minimized (we assume maximum likelihood detection and equally likely symbols). Applications involving coding are beyond the scope of this paper and will be reported elsewhere. In the following, we propose a straightforward way to carry out the mapping/demapping in practical implementations. The bits-to-symbol conversion (and *vice-versa*) is performed with the help of several coordinate systems as indicated in figure 2. These coordinate systems clearly are symmetric with respect to each other as already discussed for the *squares* which enable to construct a $(4M)$-point constellation from an $M$-point one, as discussed at the beginning of section II.

In many instances, it will be necessary to employ a coordinatewise vector product denoted by $\star$ and defined to be :

$$(4)\quad \boldsymbol{P}_1 \star \boldsymbol{P}_2 \triangleq (x_1x_2, y_1y_2) \qquad \text{were} \quad \boldsymbol{P}_i = (x_i, y_i) \in R^2,\ i = 1, 2.$$

***Bits-to-symbol mapping.***

Let us consider an $M = 2^m$-point ($m$ even) two-dimensional square constellation. A binary word $\boldsymbol{b}$ of length $m$ is assigned to each signal point as indicated in figure 1. Initially, each two consecutive bits are coupled as a 2-dimensional vector $\boldsymbol{b}_i = (b_{i1}, b_{i2})$, defining $m/2$ vectors in such a way that the binary word $\boldsymbol{b}$ can be written as :

$$(5)\qquad \boldsymbol{b} = (\boldsymbol{b}_{m/2-1}, \boldsymbol{b}_{m/2-2}, \cdots, \boldsymbol{b}_1, \boldsymbol{b}_0).$$

The conversion of the binary information into a signal point can be made according to the following procedure :

**Step 1 :**

Generate a new word $\boldsymbol{B}$ by replacing 0's by $-1$ in $\boldsymbol{b}$, thus defining :

$$(6)\qquad \boldsymbol{B} = (\boldsymbol{B}_{m/2-1}, \boldsymbol{B}_{m/2-2}, \cdots, \boldsymbol{B}_1, \boldsymbol{B}_0).$$

**Step 2 :**

The constellation point is given by :

$$(7)\ \boldsymbol{P} = \boldsymbol{B}_{m/2-1}2^{m/2-1} + \cdots + \boldsymbol{B}_{m/2-1} * \boldsymbol{B}_{m/2-2} * \cdots * \boldsymbol{B}_1 2^1 + \boldsymbol{B}_{m/2-1} * \cdots * \boldsymbol{B}_1 * \boldsymbol{B}_0 2^0,$$

where * denotes the vector product introduced in (4).

In order to illustrate this method, we shall consider the $M = 64$-point 2D constellation represented in figure 1c. As an example, let us find the constellation point $\boldsymbol{P}$ assigned to a given binary word, say (100011) : we have :

$$\boldsymbol{B} = (1\ \ -1\ \ -1\ \ -1\ \ 1\ \ 1) = (\boldsymbol{B}_2, \boldsymbol{B}_1, \boldsymbol{B}_0)$$

so that :

$$\boldsymbol{P} = \boldsymbol{B}_2 2^2 + \boldsymbol{B}_2 * \boldsymbol{B}_1 2^1 + \boldsymbol{B}_2 * \boldsymbol{B}_1 * \boldsymbol{B}_0 2^0 = (4,\ -4) + (-2,\ 2) + (-1,\ 1) = (1,\ -1).$$

The interpretation of this assignment method is quite simple. The first two bits (10) select the IV-th quadrant of the 64-point constellation so it is reduced to a 16-point one with its origin displaced to the (4,–4) point. The next two bits (00) select the III-rd quadrant in the latter constellation, so it is reduced to a 4-point constellation with its origin displaced from (–2,2) with respect to the (4,–4) point. This means that the origin was now moved to the point (–2,–2). Lastly, the bits (11) select the I-st quadrant, i.e., the (1,–1) signal point.

***Symbol-to-bits demapping.***

Let sgn(.) be an operator defined over $\mathbb{R}^2$ which indicates only the sign of the two coordinates, that is,

(8)
$$\mathrm{sgn}(\boldsymbol{P}_i) \triangleq (\mathrm{sgn} x_i,\ \mathrm{sgn} y_i),\ \text{given } \boldsymbol{P}_i = (x_i, y_i) \in \mathbb{R}^2.$$

In this section, $\boldsymbol{B}_i$ will denote the 2-dimensional vector obtained by :

(9) $$\boldsymbol{B}_i = \mathrm{sgn}(\boldsymbol{P}_i).$$

Before we start describing the demapping algorithm, we find it convenient to introduce $m/2$ special points defined by :

(10) $$\boldsymbol{C}_i \triangleq 2i(1,1) \text{ for } i = 0, 1, 2 \cdots, m/2 - 1.$$

These points are shown in the constellation drawn in figure 2.

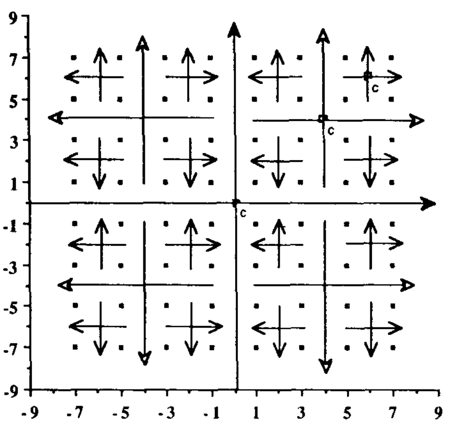


FIG. 2. — Coordinate systems which are used for both the bits-to-symbol conversion and the simple detection of received data when a square constellation and bits-to-symbol assignment of figure 1 are used.

*Systèmes de coordonnées utilisés pour la conversion bits à symbole et pour la détection simple des données reçues quand une constellation carrée et une application bits à symbole conformes à la figure 1 sont utilisées.*

Let $\boldsymbol{P}$ be the received noisy vector. A simple detection can be performed by the following algorithm :

set $i = m/2 - 1; \boldsymbol{P}_i = \boldsymbol{P}$ (initial conditions)

**step 1 :** $\boldsymbol{B}_i = \mathrm{sgn}(P_i)$;

if $i = 0$, then stop detection

**step 2 :** $\boldsymbol{P}_{i-1} = \boldsymbol{B}_i * \boldsymbol{P}_i - \boldsymbol{C}_i$;

$i = i - 1$;

go to step 1

The detected binary word $\boldsymbol{b}$ is the one which corresponds to $\boldsymbol{B}$.

An advantage of this technique in recovering digital information is that it does not require look-up tables. It is also interesting to mention that step 1 runs exactly $m/2$ times and that a partial detection of two bits occurs at each run. This implies that the information can be delivered to the sink (destination) before the end of the overall detection process, a possible advantage for some very high-speed applications.

Once again we present an illustrative example using the constellation represented in figure 1c. Assuming that the received vector is $\boldsymbol{P} = (-3.2, 4.9)$, the algorithm above results in :

$$\begin{array}{lcl} \boldsymbol{P}_2 = (-3.2, 4.9) & & \boldsymbol{B}_2 = (-1,\ 1) \\ \boldsymbol{P}_1 = (-0.8, 0.9) & \text{therefore} & \boldsymbol{B}_1 = (-1,\ 1) \\ \boldsymbol{P}_0 = (-1.2, -1.1) & & \boldsymbol{B}_0 = (-1, -1) \end{array}$$

Thus $\boldsymbol{B} = (-11\ \ -11\ \ -1-1)$ and the detected binary sequence is (010100), which is the constellation point closest to $\boldsymbol{P}$. One can easily find the pair of axes in figure 2 which is the coordinate system relevant to the points $\boldsymbol{P}_i$.

## III. DEFINITION AND PERFORMANCE OF 2D GENERALIZED CROSS CONSTELLATIONS

Let us consider the use of some multiple signal phase-amplitude schemes over an AWGN channel. It is established [10] that uncoded QAM performance with square constellations is given by :

(11)
$$P_e(2)_M = 1 - \left[1 - \left(1 - \frac{1}{\sqrt{M}}\right) \mathrm{erfc}\ \left(\sqrt{\frac{3\gamma_{\mathrm{av}}}{2(M-1)}}\right)\right]^2,$$

where $P_e(2)_M$ is the error probability per 2D symbol, $M = 2^m$ is the number of signal points, $m$ even, and $\gamma_{\mathrm{av}} = E_{\mathrm{av}}/N_0$ is the signal-to-noise ratio, $E_{\mathrm{av}}$ being the average energy per 2D symbol.

Accordingly, $\gamma_{\mathrm{av}}$ represents the ratio of the average energy to the noise spectral density, both per 2D. Dividing both the numerator and denominator by two results in the same ratio per dimension : $\gamma_{\mathrm{av}} = 2E_N/N_0$, where $E_N = E_{\mathrm{av}}/2$ is the average energy per dimension. Thus, $\gamma_{\mathrm{av}}$ is expressed in [W.s/dim]/[(W/Hz)/dim], so it is a dimensionless quantity. Moreover, assuming a signaling rate of $2B$ dim/s we find this is equivalent to the ratio $S/N$ usually found in the capacity formula. We claim this SNR measure is more suitable for channels in the bandwidth-limited region than $E_b/N_0$ which is more interesting for channels in the power-limited region.

Of course, we may neglect $1/\sqrt{M}$ in (11) if $M$ is large enough, which results in :

$$(12)\quad P_e(2)_M \leq 1 - \left[1 - \text{erfc}\left(\sqrt{\frac{3\gamma_{\text{av}}}{2(M-1)}}\right)\right]^2,$$

where the bound gets tighter and tighter as $M$ increases.

Forney and Wei defined the figure of merit of a constellation $C$ [7] as :

$$(13)\quad \text{CFM}(C) \triangleq \frac{d^2_{\min}}{P(C)},$$

where $P(C)$ is the average power of $C$. Instead, we propose to use :

$$(14)\quad \text{FM}(C) \triangleq \frac{d^2_{\min}/4}{P(C)} = \frac{\rho^2}{p(C)} = \text{CFM}(C)/4,$$

which is the multiplying factor of $\gamma_{\text{av}}$ in the error probability expressions. It can be substituted for CFM$(C)$ in the asymptotic gain formula when coded constellations are used, since :

$$(15)\quad g_{\text{asymp}} = \frac{\text{CFM}(C)_{\text{cod}}}{\text{CFM}(C)_{\text{unc}}} = \frac{\text{FM}(C)_{\text{cod}}}{\text{FM}(C)_{\text{unc}}} = \frac{(d^2_{\min})_{\text{cod}}}{(d^2_{\min})_{\text{unc}}} \frac{P(C)_{\text{unc}}}{P(C)_{\text{cod}}}.$$

This figure of merit can be extended to lattice coded systems [11]. We therefore prefer to consider normalized constellations with respect to $d_{\min} = 2$ whereas Forney and Wei use $d_{\min} = 1$ as normalized value (our choice corresponds to $\rho = 1$). Hence, the normalized average power values printed in our tables have been calculated for constellations where $d_{\min} = 2$, i.e., the signal points were taken in an odd-integer $2Z^2 + (1,1)$ lattice.

Ordinarily, the performance of QAM systems is analysed at high signal-to-noise ratios ($\gamma_{\text{av}} \gg 2(M-1)/3$), in which case the main term in (12) results in a reasonable approximation of the symbol error rate, namely :

$$(16)\quad P_e(2)_M \approx 2\text{erfc}\left(\sqrt{\frac{3\gamma_{\text{av}}}{2(M-1)}}\right) \approx \exp[-\text{FM}(C)\gamma_{\text{av}}].$$

Let us now introduce a family of constellations to be referred to as 2D *generalized cross constellations*, which are defined as the $M$-point least-energy constellations ($M$ a multiple of 4) whose points belong to the odd-integer grid $2Z^2 + (1,1)$ and which furthermore satisfy a symmetry constraint. This constraint may be :

— bilateral symmetry, when the constellation is assumed to be symmetric with respect to the coordinate axes;

— rotational symmetry, when the constellation is assumed to be invariant by a $\pi/2$ rotation;

— and quadrilateral symmetry, when the constellation is both bilaterally and rotationally symmetric.

Figure 3 illustrates such symmetries in the case of $M = 8$.

The odd-integer grid $2Z^2 + (1,1)$ may be partitioned into *shells* defined as the set of all points of the grid having the same energy, hence which lie at a same distance from the origin. All clearly possess quadrilateral symmetry. Then we can satisfy the least energy constraint by taking first the shells of smallest energy (provided the symmetry constraint can be satisfied), and the chosen symmetry constraint by dividing the shells, if necessary, only in subsets having themselves the chosen symmetry. Such generalized cross constellations can thus be designed using Table I, which lists the shells of the odd-integer grid, ordered by increasing energy.

This definition can be extended to $M$ non multiple of 4 by appending 1, 2 or 3 points belonging to the next shell (or to the remaining part of the last shell if it was only partially used) to one of the previous constellations with $M$ a multiple of 4.

Figure 4 shows some of the generalized constellations with $M$ a multiple of 4 and quadrilateral symmetry. It can be noticed that all $(2k)^2$-point constellations of this family are square signal sets for $k$ up to 4. It can also be noticed that for several constellations e.g., the 8- and 28-point ones, a 4-point shell cannot be included in the constellation due to the symmetry constraint, thus leaving *holes*. Parameters of some such generalized cross constellations are given in Table II, together with those of the least-energy constellations belonging to the same grid (but for which the constraint of quadrilateral symmetry was relaxed). These parameters are the number of points $M$, the constellation power $P(C)$, its expression in decibels, and the peak-to-average power

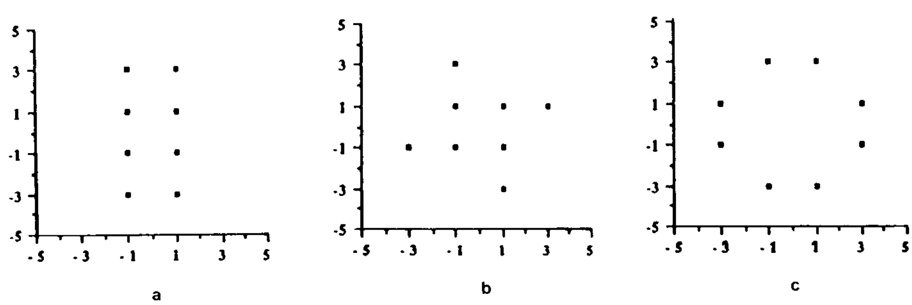


FIG. 3. — 8-point constellation with different symmetries : bilateral (3a), rotational (3b) and quadrilateral (3c).

*Constellation à 8 points avec différentes symétries : bilatérale (3a), de rotation (3b) et quadrilatérale (3c).*

TABLE I. — Energy distribution in the $2Z^2+(1,1)$ lattice.

| Shell | Number of points | Energy per 2D | Shell | Number of points | Energy per 2D |
|---|---|---|---|---|---|
| 1 | 4 | 2 | 33 | 8 | 362 |
| 2 | 8 | 10 | 34 | 16 | 370 |
| 3 | 4 | 18 | 35 | 8 | 386 |
| 4 | 8 | 26 | 36 | 8 | 394 |
| 5 | 8 | 34 | 37 | 16 | 410 |
| 6 | 12 | 50 | 38 | 16 | 442 |
| 7 | 8 | 58 | 39 | 12 | 450 |
| 8 | 8 | 74 | 40 | 8 | 458 |
| 9 | 8 | 82 | 41 | 8 | 466 |
| 10 | 8 | 90 | 42 | 8 | 482 |
| 11 | 4 | 98 | 43 | 8 | 490 |
| 12 | 8 | 106 | 44 | 8 | 514 |
| 13 | 8 | 122 | 45 | 8 | 522 |
| 14 | 16 | 130 | 46 | 16 | 530 |
| 15 | 8 | 146 | 47 | 8 | 538 |
| 16 | 4 | 162 | 48 | 8 | 554 |
| 17 | 16 | 170 | 49 | 8 | 562 |
| 18 | 8 | 178 | 50 | 12 | 578 |
| 19 | 8 | 194 | 51 | 8 | 586 |
| 20 | 8 | 202 | 52 | 16 | 610 |
| 21 | 8 | 218 | 53 | 8 | 626 |
| 22 | 8 | 226 | 54 | 8 | 634 |
| 23 | 8 | 234 | 55 | 24 | 650 |
| 24 | 4 | 242 | 56 | 8 | 666 |
| 25 | 16 | 250 | 57 | 8 | 674 |
| 26 | 8 | 274 | 58 | 8 | 698 |
| 27 | 16 | 290 | 59 | 8 | 706 |
| 28 | 8 | 298 | 60 | 4 | 722 |
| 29 | 8 | 306 | 61 | 16 | 730 |
| 30 | 8 | 314 | 62 | 8 | 738 |
| 31 | 12 | 338 | 63 | 8 | 746 |
| 32 | 8 | 346 | 64 | 8 | 754 |

ratio PAR. The constellations of minimum energy for the same number $M$ of points have normalized power denoted by $P(C)_{\min}$ which is easily computed by taking the $M$ least-energy points in Table I. The generalized cross constellations with quadrilateral symmetry are not always identical to least energy ones, which means that they are not always optimal, but they are attractive at least from a theoretical viewpoint.

Generally, a signal set is expressed by :

$$(17) \quad s_i(t) = A_{ci}u(t)\cos(2\pi f_c t) - A_{si}u(t)\sin(2\pi f_c t), \quad i = 1, 2, \cdots, M,$$

where $u(t)$ is a narrow-band waveform, $(A_{ci}, A_{si})$ are coordinates in the 2D constellation and $f_c$ is the carrier frequency.

Assuming a signaling rate of $1/T$ bauds, an information symbol is transmitted each $T$ seconds and the QAM signal $s(t)$ is given by :

$$(18) \quad s(t) = \sum_k a_{ck}u(t-kT)\cos(2\pi f_c t) - \sum_k a_{sk}u(t-kT)\sin(2\pi f_c t),$$

where $(a_{ck}, a_{sk})$ corresponds to the $k$-th transmitted signal $(A_{ci(k)}, A_{si(k)})$.

The signal $s(t)$ can also be written in terms of its complex envelope :

$$(19) \quad s(t) = \mathrm{Re}\sum_k \boldsymbol{P}_k u(t-kT)\exp(\mathrm{j}2\pi f_c t),$$

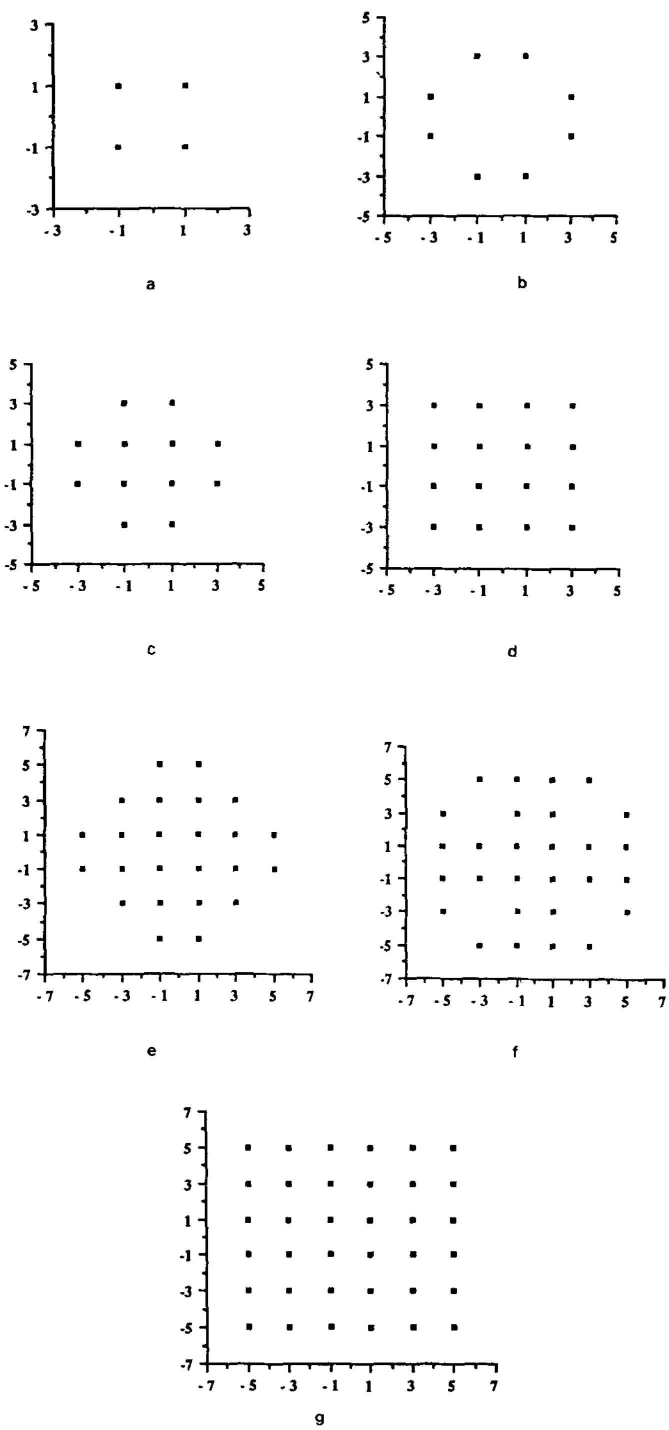


FIG. 4. — Generalized cross constellations with quadrilateral symmetry. Their sizes are 4 (4a), 8 (4b), 12 (4c), 16 (4d), 24 (4e), 28 (4f) and 36 (4g).

*Constellations en croix généralisées avec la symétrie quadrilatérale, de taille 4 (4a), 8 (4b), 12 (4c), 16 (4d), 24 (4e), 28 (4f) et 36 (4g).*

where the complex $\boldsymbol{P}_k \triangleq a_{ck} + \mathrm{j}a_{sk}$ represents a constellation point.

Each signal $s_i(t)$ is associated with a 2D point $\boldsymbol{P}_i = A_{ci} + jA_{si} = (A_{ci}, A_{si})$. The energy of a constellation point is $|\boldsymbol{P}_i|^2 = A_{ci}^2 + A_{si}^2$ and the average energy per 2D-point is $E(C) \triangleq E(|\boldsymbol{P}_i|^2) = E(A_{ci}^2 + A_{si}^2)$. The average energy $E(C)$ is often expressed as a normalized power $P(C)$ i.e., $T$ has been normalized to unity so both terms are considered interchangeable.

TABLE II. — Parameters of some generalized cross constellations.

| $M$ | $P(C)$ | dB | PAR | $P(C)_{\min}$ | dB | PAR |
|---|---|---|---|---|---|---|
| 4 | 2.0000 | 3.0103 | 1.0000 | 2.0000 | 3.0103 | 1.0000 |
| 8 | 10.0000 | 10.0000 | 1.0000 | 6.0000 | 7.7815 | 1.6666 |
| 12 | 7.3333 | 8.6530 | 1.3636 | 7.3333 | 8.6530 | 1.3636 |
| 16 | 10.0000 | 10.0000 | 1.8000 | 10.0000 | 10.0000 | 1.8000 |
| 20 | 14.8000 | 11.7026 | 1.7567 | 13.2000 | 11.2057 | 1.9696 |
| 24 | 15.3333 | 11.8564 | 1.6956 | 15.3333 | 11.8564 | 1.6956 |
| 28 | 20.2857 | 13.0719 | 1.6760 | 18.0000 | 12.5527 | 1.8888 |
| 32 | 20.0000 | 13.0103 | 1.7000 | 20.0000 | 13.0103 | 1.7000 |
| 36 | 23.3333 | 13.6798 | 2.1428 | 23.3333 | 13.6798 | 2.1428 |
| 40 | 26.0000 | 14.1497 | 1.9230 | 26.0000 | 14.1497 | 1.9230 |
| 44 | 28.1818 | 14.4997 | 1.7741 | 28.1818 | 14.4997 | 1.7741 |
| 48 | 31.3333 | 14.9601 | 1.8510 | 30.6666 | 14.8667 | 1.8913 |
| 52 | 32.7692 | 15.1547 | 1.7699 | 32.7692 | 15.1547 | 1.7699 |
| 56 | 37.4285 | 15.7320 | 1.9770 | 35.7143 | 15.5284 | 2.0720 |
| 60 | 38.2667 | 15.8282 | 1.9337 | 38.2667 | 15.8282 | 1.9338 |
| 64 | 42.0000 | 16.2325 | 2.3333 | 41.0000 | 16.1278 | 2.0000 |
| 68 | 43.4118 | 16.3761 | 1.8888 | 43.4118 | 16.3761 | 1.8888 |
| 72 | 46.4444 | 16.6693 | 2.1100 | 46.0000 | 16.6276 | 1.9565 |
| 76 | 48.3157 | 16.8409 | 1.8627 | 48.3158 | 16.8409 | 1.8627 |
| 80 | 50.8000 | 17.0586 | 1.9291 | 50.8000 | 17.0586 | 1.9291 |
| 84 | 53.8095 | 17.3086 | 1.9699 | 53.4286 | 17.2777 | 1.9839 |
| 88 | 55.8182 | 17.4678 | 1.8990 | 55.8182 | 17.4678 | 1.8990 |
| 92 | 60.4347 | 17.8129 | 2.1510 | 58.6957 | 17.6861 | 2.0785 |
| 96 | 62.0000 | 17.9239 | 2.0967 | 61.3333 | 17.8770 | 1.9891 |
| 100 | 66.0000 | 18.1954 | 2.4545 | 64.0800 | 18.0672 | 2.0287 |
| 104 | 66.6153 | 18.2357 | 1.9515 | 66.6154 | 18.2357 | 1.9515 |
| 108 | 70.1481 | 18.4602 | 2.3093 | 68.9630 | 18.3862 | 1.8850 |
| 112 | 71.1429 | 18.5213 | 1.8273 | 71.1429 | 18.5213 | 1.8273 |
| 116 | 74.2758 | 18.7085 | 2.1810 | 73.7241 | 18.6761 | 1.9803 |
| 120 | 76.1333 | 18.8157 | 1.9176 | 76.1333 | 18.8157 | 1.9176 |
| 124 | 78.9032 | 18.9709 | 2.0531 | 78.9032 | 18.9709 | 2.0531 |
| 128 | 82.0000 | 19.1381 | 2.0731 | 81.7500 | 19.1249 | 2.0795 |
| 132 | 84.4242 | 19.2647 | 2.0136 | 84.4242 | 19.2647 | 2.0136 |
| 136 | 89.0588 | 19.4968 | 2.2681 | 86.9412 | 19.3923 | 1.9555 |
| 140 | 91.1428 | 19.5972 | 2.2163 | 89.3143 | 19.5092 | 1.9033 |
| 144 | 95.3333 | 19.7924 | 2.5384 | 91.7777 | 19.6274 | 1.9394 |
| 148 | 95.4054 | 19.7957 | 2.1172 | 94.1081 | 19.7363 | 1.8914 |
| 152 | 99.2631 | 19.9679 | 2.4379 | 96.7368 | 19.8559 | 2.0054 |
| 156 | 99.6410 | 19.9844 | 2.0272 | 99.2308 | 19.9665 | 1.9550 |
| 160 | 103.200 | 20.1368 | 2.3449 | 101.800 | 20.0775 | 1.9842 |
| 164 | 104.244 | 20.1805 | 1.9377 | 104.244 | 20.1805 | 1.9377 |
| 168 | 107.523 | 20.3150 | 2.2506 | 106.952 | 20.2919 | 2.0382 |
| 172 | 109.535 | 20.3955 | 1.9902 | 109.535 | 20.3955 | 1.9902 |
| 176 | 112.545 | 20.5133 | 2.1502 | 112.182 | 20.4992 | 2.0145 |
| 180 | 115.777 | 20.6363 | 2.1593 | 114.711 | 20.5961 | 1.9701 |
| 184 | 118.521 | 20.7380 | 2.1093 | 117.304 | 20.6931 | 1.9948 |
| 188 | 123.191 | 20.9058 | 2.3540 | 119.787 | 20.7841 | 1.9534 |
| 192 | 125.666 | 20.9922 | 2.3076 | 122.333 | 20.8754 | 1.9782 |
| 196 | 130.000 | 21.1394 | 2.6000 | 124.939 | 20.9670 | 2.0009 |
| 200 | 129.680 | 21.1287 | 2.2362 | 127.440 | 21.0531 | 1.9617 |
| 204 | 133.764 | 21.2634 | 2.5268 | 129.843 | 21.1342 | 1.9254 |
| 208 | 133.692 | 21.2611 | 2.1691 | 132.154 | 21.2108 | 1.8917 |
| 212 | 137.547 | 21.3845 | 2.4573 | 134.830 | 21.2979 | 2.0321 |
| 216 | 138.000 | 21.3988 | 2.1014 | 137.407 | 21.3801 | 1.9940 |
| 220 | 141.636 | 21.5117 | 2.3863 | 140.182 | 21.4669 | 2.0687 |
| 224 | 142.857 | 21.5490 | 2.0300 | 142.857 | 21.5490 | 2.0300 |
| 228 | 146.280 | 21.6519 | 2.3106 | 145.439 | 21.6268 | 1.9939 |
| 232 | 148.482 | 21.7168 | 2.0608 | 147.931 | 21.7006 | 1.9603 |
| 236 | 151.694 | 21.8097 | 2.2281 | 150.475 | 21.7746 | 1.9804 |
| 240 | 155.066 | 21.9052 | 2.2312 | 152.933 | 21.8450 | 1.9485 |
| 244 | 158.065 | 21.9884 | 2.1889 | 155.443 | 21.9157 | 1.9685 |
| 248 | 162.774 | 22.1159 | 2.4205 | 157.871 | 21.9830 | 1.9382 |
| 252 | 165.555 | 22.1894 | 2.3798 | 160.349 | 22.0507 | 1.9582 |
| 256 | 170.000 | 22.3045 | 2.6470 | 162.750 | 22.1152 | 1.9293 |

It is supposed that the signals on phase and quadrature carriers can perfectly be separated by coherent detection, and that maximum likelihood detection is carried out by the receiver.

Let $P_e(i\text{-th D})$ denote the probability of an erroneous decision in the $i$-th dimension, $i = 1, 2$. For simplicity, we shall consider constellations with quadrilateral symmetry such that :

(20) $$P_e(1-\text{st D}) = P_e(2\text{–nd D}) \triangleq P_e(1)_M,$$

where $P_e(1)_M$ is the error probability per dimension for the $M$-ary QAM system. Generally, detection must take account of all the signal dimensions because there is some dependence between signal coordinates (except for square constellations).

Let $P_e(1 \mid \boldsymbol{P}_i)$ denote the conditional probability of an erroneous decision per dimension given that the signal point $\boldsymbol{P}_i$ was transmitted. Obviously, $P_e(1)_M$ can be evaluated as :

(21) $$P_e(1)_M = \sum_{i=0}^{M-1} P(\boldsymbol{P}_i) P_e(1 \mid \boldsymbol{P}_i),$$

where $P(\boldsymbol{P}_i)$ is the probability of $\boldsymbol{P}_i$ to be sent.

In an unbounded constellation, all the signal points have the same decision region so $P_e(1 \mid \boldsymbol{P}_i)$ does not depend on the taken point, that is, $P_e(1 \mid \boldsymbol{P}_i) = p_\varepsilon$. Substituting this in (21), it follows that $P_e(1) = p_\varepsilon$ for the boundless constellation ($M \to \infty$). For bounded constellations, this is an upper bound on the one-dimensional error rate (since $P_e(1 \mid \boldsymbol{P}_i) \leq p_\varepsilon$), that is :

(22) $$P_e(1)_M \leq P_e(1)_\infty = p_\varepsilon.$$

Likewise, the 2D error rate (symbol error probability), $P_e(2)_M$, is also upper bounded by the symbol error probability for the unbounded constellation. In this case, all points have identical Voronoi regions so that :

(23) $$P_e(2)_M \leq P_e(2)_\infty = 1 - [1 - p_\varepsilon]^2,$$

since the noises added to both components are statistically independent. Therefore, the error rate of such a signal set is essentially controlled by $p_\varepsilon$.

We shall now investigate the error probability per dimension in an unbounded constellation over an ideal band-limited AWGN channel.

**Theorem 1.**

The one-dimension error probability $p_\varepsilon$ of $M$-ary QAM systems based on 2D generalized uncoded constellations (border effect neglected) over an AWGN channel is given by $p_\varepsilon = \mathrm{erfc}(\sqrt{\text{FM}(C)\gamma_{\text{av}}})$ ∎

***Proof.***

The proof is straightforward and is based on a number of known results.

Defining the symbol autocorrelation function as $R_P(i) \triangleq E(\boldsymbol{P}_k \boldsymbol{P}^*_{k-i})$, it can be shown that the spectral power density of the signal $s(t)$ is given by (provided the quadrilateral symmetry) :

(24)
$$S_s(f) = \frac{1}{4T} R_P(0) [\mid U(f - f_c) \mid^2 + \mid U(-f - f_c) \mid^2],$$

where $U(f)$ is the Fourier transform of $u(t)$ introduced in (18). Of course, $R_P(0) = E(\mid \boldsymbol{P}_i \mid^2) = E(C)$.

The average power $P_{\text{av}}$ of the QAM signal is found by integrating (24), that is,

$$(25)\quad P_{\text{av}} = \frac{E(C)}{4T}\int_{-\infty}^{+\infty}[\mid U(f-f_c)\mid^2 + \mid U(-f-f_c)\mid^2]\mathrm{d}f = \frac{E(C)}{2T}\xi_u,$$

where $\xi_u = \int_{-\infty}^{+\infty} \mid u(t)\mid^2 \mathrm{d}t = \int_{-\infty}^{+\infty}\mid U(f)\mid^2 \mathrm{d}f$ is the energy of the narrow-band pulse $u(t)$.

Hence, the average energy of the QAM signal is :

$$(26)\qquad E_{\text{av}} = P_{\text{av}}T = E(C)\xi_u/2.$$

As explained above, we are interested in the signal-to-noise ratio :

$$(27)\qquad \gamma_{\text{av}} = \frac{2E_N}{N_0} = P(C)\frac{\xi_u}{2N_0}.$$

We shall use the pass-band representation [12] of the random noise signal $n_w(t)$ in a bandwidth $B$ Hz. The noise is assumed to be *white* with spectral power density :

$$(28)\qquad S_{n_w}(f) = \begin{cases} N_0/2, & |f| < B \\ 0, & \text{otherwise.}\end{cases}$$

The narrow-band noise waveform can be expressed as :

$$(29)\quad n_w(t) = n_c(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t),$$

where $n_c(t)$ and $n_s(t)$ are both low-pass noise signals band-limited to $B/2$ having as power spectral density

$$(30)\qquad S_{n_c}(f) = S_{n_s}(f) = \begin{cases} N_0, & |f| < B/2, \\ 0, & \text{otherwise.}\end{cases}$$

Since the noise is assumed to be *additive*, the received signal is $s(t) + n_w(t)$ where the signal and the noise are given by (18) and (29), respectively. The received waveform is then processed : a coherent detection is performed followed by matched filtering and decision-making.

In order to achieve the maximum transmission rate we need use the fastest signaling rate. We know the bound $2B$ dim/s on the number of dimension per second accommodated by a band-limited channel [2]. In the case of 2D constellations, a symbol corresponds to 2 dimensions so the transmission speed must be less than $B$ bauds $= B$ symbols per second.

On the other hand, Nyquist's criterion asserts that a bandwidth $1/2T$ (base band) or $1/T$ (bandpass) is the minimum bandwidth needed for transmitting without intersymbol interference (ISI) at a rate $1/T$ symbol per second. These two approaches confirm that $T = 1/B$ represents the minimum interval where we can transmit without intersymbol interference on a band-limited channel of bandwidth $B$ Hz.

The equivalent low-pass signals at the receiver can be written as :

$$(31a)\qquad r_c(t) = \sum_k a_{ck}u(t-kT) + n_c(t)$$

$$(31b)\qquad r_s(t) = \sum_k a_{sk}u(t-kT) + n_s(t).$$

Hereafter, the index $c$ or $s$ will be dropped because the signal detection is identical in both the quadrature and phase components. We write simply :

$$(32)\qquad r(t) = \sum_k a_k u(t-kT) + n(t).$$

The impulse response of the matched filter is $h(t) = u^*(-t)$, so that its output is :

$$(33)\quad y(t) = \int_{-\infty}^{+\infty} r(\alpha)h(t-\alpha)\mathrm{d}\alpha = \sum_k a_k\int_{-\infty}^{+\infty} u(\alpha-kT)u^*(\alpha-t)\mathrm{d}\alpha + \int_{-\infty}^{+\infty} n(\alpha)u^*(\alpha-t)\mathrm{d}\alpha.$$

The output of the matched filter at the sampling instant $nT$ is the decision variable $y_n$ :

$$(34)\quad y_n = \sum_k a_k\int_{-\infty}^{+\infty} u(\alpha-kT)u^*(\alpha-nT)\mathrm{d}\alpha + \int_{-\infty}^{+\infty} n(\alpha)u^*(\alpha-nT)\mathrm{d}\alpha.$$

It is known [12] that the response of a linear filter to a random Gaussian process is a Gaussian random variable, so :

$$(35)\quad w_n \triangleq \int_{-\infty}^{+\infty} n(\alpha)u^*(\alpha-nT)\mathrm{d}\alpha \propto N(0, N_0\xi_u).$$

On the other hand, defining $p(t) \triangleq u(t)\cdot u^*(-t)$, it follows that :

$$(36)\qquad y_n = a_n p(0) + \sum_{k,k\neq n} a_k p(nT-kT) + w_n.$$

The first term is the desired signal while the second and the third ones represent the intersymbol interference and additive Gaussian noise, respectively.

We assume $u(t)$ real and such that the waveform $p(t)$ obeys the first Nyquist criterion so the ISI vanishes. Therefore, we meet here the well known equally split filtering between the transmitter and the receiver.

Finally, we have :

$$(37)\qquad y_n = a_n\xi_u + w_n.$$

Alternatively, we can consider a *normalized* decision value $Y_n \triangleq a_n + W_n$, where $W_n \propto N(0, \sigma^2 \triangleq N_0/\xi_u)$. It is trivially verified that the one-dimensional error probability $p_\varepsilon$ is :

$$(38)\qquad p_\varepsilon = \operatorname{erfc}\left(\sqrt{\frac{d_{\min}^2/4}{2\sigma^2}}\right).$$

Keeping Forney's notation, we rewrite (38) as :

$$(39)\quad p_\varepsilon = \operatorname{erfc}(\sqrt{(d_{\min}^2/4)\gamma_{\text{av}}/P(C)}) = \operatorname{erfc}(\sqrt{\text{FM}(C)\gamma_{\text{av}}}),$$

QED.

A more accurate estimate of the error probability must take into account the boundedness of the constellation. The exact performance calculation is complicated due to the shape of Voronoi regions concerning the outmost points. The reduction in the estimate of $P_e(2)_M$ depends on the number of outmost points and can be obtained as follows :

**Theorem 2.**

The performance of $M$-ary QAM systems with symmetrical generalized cross constellations satisfies the bounds $2\alpha_L p_\varepsilon - \beta_L p_\varepsilon^2 \leq P_e(2)_M \leq 2\alpha_U p_\varepsilon - \beta_U p_\varepsilon^2$, where $p_\varepsilon$ is the one-dimensional error probability given by Theorem 1 and $\alpha$, $\beta$ are positive real numbers less than unity defined below. ∎

***Proof.***

Let $\Gamma_I$ (resp. $\Gamma_0$) be the set of all inner (resp. outer) points and let $|\Gamma|$ denote the cardinality of the set $\Gamma$. Two kinds of outmost points, namely type I and II, can be considered. First, the points that are outmost regarding just one dimension, and secondly those which are outmost regarding both dimensions. Consequently the set of outmost points, $\Gamma_O$, can be partitioned into two disjoint subsets, $\Gamma_O^I \cup \Gamma_O^{II}$. We neglect in this analysis the presence of a few inner points which are not completely surrounded by other ones, which occurs in a few constellations.

We shall suppose henceforth, as usual, that all signal points are equally likely *a priori*. Thus : $P(\Gamma) = |\Gamma|/M$. Certainly $|\Gamma_I| + |\Gamma_O| = M$, so that $P(\Gamma_I) + P(\Gamma_O) = 1$.

In order to circumvent the difficulty associated with the dependence between signal coordinates, we have bounded the error rate by taking rectangular decision regions which are sometimes different from the Voronoi regions. The following relations are then established :

(40a) $$P_e(2|\boldsymbol{P}_i \in \Gamma_I) = 1 - [1 - p_\varepsilon]^2$$

(40b) $$1 - [1 - p_\varepsilon][1 - p_\varepsilon/2] \leq P_e(2|\boldsymbol{P}_i \in \Gamma_O^I) \leq 1 - [1 - p_\varepsilon]^2$$

(40c) $$1 - [1 - p_\varepsilon/2]^2 \leq P_e(2|\boldsymbol{P}_i \in \Gamma_O^{II}) \leq 1 - [1 - p_\varepsilon][1 - p_\varepsilon/2].$$

The symbol error probability of QAM systems can clearly be written as :

(41) $$P_e(2)_M = \sum_{\boldsymbol{P}_i \in \Gamma_I} P(\boldsymbol{P}_i) P_e(2|\boldsymbol{P}_i) + \sum_{\boldsymbol{P}_i \in \Gamma_O^I} P(\boldsymbol{P}_i) P_e(2|\boldsymbol{P}_i) + \sum_{\boldsymbol{P}_i \in \Gamma_O^{II}} P(\boldsymbol{P}_i) P_e(2|\boldsymbol{P}_i).$$

Applying relations (40) to the above equality, the following bound is obtained :

(42) $$P_e(2)_M \geq 1 - [P(\Gamma_I)[1 - p_\varepsilon]^2 + P(\Gamma_O^I)[1 - p_\varepsilon][1 - p_\varepsilon/2] + P(\Gamma_O^{II})[1 - p_\varepsilon/2]^2],$$

which after some computation yields the lower bound :

(43) $$P_e(2)_M \geq 2\alpha_L p_\varepsilon - \beta_L p_\varepsilon^2,$$

where $\alpha_L = P(\Gamma_I) + 0.75P(\Gamma_O^I) + 0.5P(\Gamma_O^{II})$ and $\beta_L = P(\Gamma_I) + 0.5P(\Gamma_O^I) + 0.25P(\Gamma_O^{II})$.

Furthermore, we have :

(44) $$P_e(2)_M \leq 1 - [P(\Gamma_I)[1 - p_\varepsilon]^2 + P(\Gamma_O^I)[1 - p_\varepsilon]^2 + P(\Gamma_O^{II})[1 - p_\varepsilon][1 - p_\varepsilon/2]],$$

yielding the following upper bound :

(45) $$P_e(2)_M \leq 2\alpha_U p_\varepsilon - \beta_U p_\varepsilon^2,$$

where $\alpha_U = P(\Gamma_I) + P(\Gamma_O^I) + 0.75P(\Gamma_O^{II})$ and $\beta_U = P(T_I) + P(\Gamma_O^I) + 0.5P(\Gamma_O^{II})$. QED ∎

The performance evaluation of generalized cross constellations by Theorem 1 or 2 needs the normalized average power $P(C)$ of the constellations, which can be found in Table II.

It can be seen that $M$ approaching infinity (i.e., an unbounded constellation) implies $\alpha_L$, $\alpha_U \to 1$ and $\beta_L$, $\beta_U \to 1$, since the outer points become a vanishingly small fraction of the total number of points. Therefore the upper and lower bounds agree in the limit and $P_e(2)_M$ reduces to (23) as expected. Furthermore, an $M = 2^m$-point ($m$ even) square constellation can be found for which :

$$P(\Gamma_O^{II}) = 4/M; \; P(\Gamma_I) = 1 - \frac{4}{M}(\sqrt{M} - 1)$$

and

$$P(\Gamma_O^I) = \frac{4}{M}(\sqrt{M} - 2)$$

so that

$$\alpha_L = \beta_L^2 = 1 - \frac{1}{\sqrt{M}}$$

and the lower bound is reached (see (11)).

It can be conjectured that the symbol error probability of any 2D cross constellation is well fitted by a curve $P_e(2)_M = 2\alpha p_\varepsilon - \beta p_\varepsilon^2$ for some $\alpha$, $\beta$ both less than, but close to, unity.

In order to check the validity of these results, the performance of a 12-ary QAM system in the presence of AWGN (generated by the polar method [13]) was evaluated by Monte Carlo simulation. Figure 5 represents computer simulation results of the symbol error rate versus

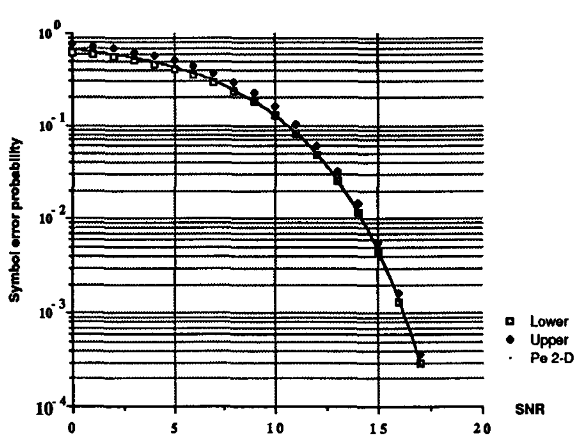


FIG 5. — Bounds on the symbol error probability for the 12-point constellation of figure 4c and the corresponding simulated error rate.

*Bornes sur la probabilité d'erreur par symbole pour la constellation à 12 points de la figure 4c et taux d'erreur correspondant simulé.*

signal-to-noise ratio. The least square regression analysis shows that the symbol error rate fits the $2\ \alpha p_\varepsilon - \beta p_\varepsilon^2$ curve, where $\alpha = 0.706$ and $\beta = 0.475$, with a correlation coefficient $r = 0.999$ and a standard error of estimation $s = 0.013$ for $n = 18$ data points. The upper and lower bounds were also drawn for comparison (solid lines). It can be seen that the simulated performance closely agrees with the results predicted from Theorem 2.

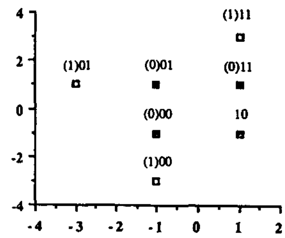


FIG. 6. — 7-point generalized cross-constellation with opportunistic channel. The basic constellation is the square 4-point one with shaded points. 3 more points are appended to 3 of its points and bear the same 2-bit label. A third *opportunistic* bit (in brackets) is appended to this label to tell if the corresponding point belongs or not to the basic constellation.

*Constellation en croix généralisée avec canal opportuniste. La constellation de base est la constellation carrée à 4 points en grisé. 3 points supplémentaires sont adjoints à 3 d'entre eux et portent la même étiquette à 2 bits. Un troisième bit* opportuniste *(entre parenthèses) est adjoint à cette étiquette pour indiquer si le point correspondant appartient ou non à la constellation de base.*

## IV. OPPORTUNISTIC SECONDARY CHANNEL

The generalized 2D cross constellations described in the previous section can be regarded as a way to convey a non-integral number of bits per symbol (or bits per 2D). In a recent paper [7], Forney and Wei discussed the use of generalized multidimensional constellations and their ability of supporting an opportunistic secondary channel. In earlier applications, this technique was conceived as a means to create an opportunity to transmit additional side information (like internal signaling and control data) [14]. More generally, it can be used for increasing the information bit rate by a small fractional number, i.e., as a way to introduce an additional rate in generalized constellations. We intend here to make a precise estimate of this potential additional normalized rate.

Given an $M$-point 2D generalized cross constellation ($M$ different from a square), the corresponding basic constellation is defined as the largest $2^m$-point constellation of the same family which is contained in it. Each point of the initial $M$-point constellation which does not belong to the basic constellation is associated with a point which belongs to it. These points bear both the same label, except that an *opportunistic bit* is appended to this label, which takes value 0 or 1 according to whether it belongs or not to the basic constellation, respectively. Of course, since the points outside the basic constellation are strictly less numerous than those which belong to it, some points of the basic constellation do not bear an opportunistic bit. If the transmitted signal corresponds to one of these points, we say that the direct channel was used; otherwise, we say that the opportunistic channel was selected.

For the sake of simplicity, we show as an example the 7-point constellation drawn in Figure 6. The basic constellation (shaded points) has only 4 points as in Figure 1a. The opportunistic bit has been written into brackets. Assuming the source bits to be independent and equally likely, we assume that the next 2 bits to be transmitted are examined in order to determine if the opportunistic channel can be used. Therefore, we have in this example :

- $P$(select opportunistic channel) $= P(\{00, 01, 11\}) = 3/4$,
- $P$(select direct channel) $= P(\{10\}) = 1/4$.

If the direct channel is employed, only the 2 information bits examined are transmitted. Otherwise, the opportunistic channel is selected so the next bit can be also transmitted and 3 information bits are conveyed. Thus, the average transmitted number of information bits per two-dimension signal is $\beta = \frac{1}{4} \cdot 2 + \frac{3}{4} \cdot 3 = 2.75$ bits/2D.

In the general case, $m$ is clearly equal to $m = \lfloor \log_2 M \rfloor$, where the floor function $\lfloor x \rfloor$ denotes as usual the largest integer no greater than $x$. Of course, an integer $W$ exists, $0 \leq W < 2^m$, such that :

$$(46) \qquad M = 2^m + W,$$

where $W$ represents the number of points appended to the basic constellation. The use of the opportunistic secondary channel thus results in transmitting an amount of $m$ bits per 2D symbol when the direct channel is used; in contrast, $m + 1$ bits (per 2D symbol) are transmitted when the opportunistic channel is selected.

Let $P_{\text{dir}}$ and $P_{\text{opp}}$ be the probability of selecting the direct channel and the opportunistic channel, respectively. The average transmission rate (in information bits per 2D) is given by :

$$(47) \qquad \beta = P_{\text{dir}} m + P_{\text{opp}}(m + 1) \quad \text{bits/2D}.$$

For independent and equally likely source bits, it follows that :

$$(48) \qquad \beta = \frac{2^m - W}{2^m} m + \frac{W}{2^m}(m + 1) = m + \frac{W}{2^m}.$$

Since $0 \leq W < 2^m$, we can use the binary decomposition to write $W = \sum_{i=0}^{m-1} d_i 2^i$. Defining $d_m \triangleq m$, we may rewrite the normalized rate $\beta$ in bits per 2-dimension as :

$$(49) \qquad \beta = \sum_{i=0}^{m} \frac{d_i 2^i}{2^m}.$$

Now, we shall investigate an explicit relationship between the normalized bit rate $\beta$ and the constellation size $M$. To begin with, let us consider a new function $(\log)_2^\dagger$ to be referred to as *rational logarithm* and defined for any positive integer $M$ by :

$$(50) \qquad (\log)_2^\dagger M \triangleq \lfloor \log_2 M \rfloor + \frac{M}{2^{\lfloor \log_2 M \rfloor}} - 1.$$

Combining equations (48) and (46) and remembering that $m = \lfloor \log_2 M \rfloor$, it follows that :

$$(51) \qquad \beta = (\log)_2^\dagger M.$$

Bounds on the $(\log)_2^\dagger$ function are given by the following theorem :

**Theorem 3.**

For any integer $M > 0$, the $(\log)_2^\dagger$ function verifies :

$$(52) \qquad \lfloor \log_2 M \rfloor \le (\log)_2^\dagger M \le \log_2 M,$$

with equality iff $M = 2^m$ for some integer $m$. ∎

***Proof.***

The proof of the lower bound in (52) is immediate. From definition (50), we have :

$$(53) \quad (\log)_2^\dagger M - \lfloor \log_2 M \rfloor = \frac{M}{2^{\lfloor \log_2 M \rfloor}} - 1 = 2^{\log_2 M - \lfloor \log_2 M \rfloor} - 1 \ge 0.$$

The upper bound can be proved in the following way : from definition (50), we have :

$$(54) \quad \log_2 M - (\log)_2^\dagger M = \log_2 M - \lfloor \log_2 M \rfloor + 1 - 2^{\log_2 M - \lfloor \log_2 M \rfloor}.$$

On the other hand, one has $\lfloor \log_2 M \rfloor \le \log_2 M < \lfloor \log_2 M \rfloor + 1$, so that :

$$(55) \qquad 0 \le \log_2 M - \lfloor \log_2 M \rfloor < 1.$$

The real function $f(x) \triangleq x + 1 - 2^x$, $x$ real, satisfies the inequality :

$$(56) \qquad f(x) \ge 0 \text{ for } 0 \le x < 1.$$

Therefore, letting $x = \log_2 M - \lfloor \log_2 M \rfloor$, the claimed inequality $\log_2 M - (\log)_2^\dagger M \ge 0$ follows from (56) and (54). At last, we mention it is trivial to verify the conditions *iff* about equalities. QED ∎

Having evaluated the normalized rate $\beta$ in bits/2D supported by the 2D signaling schemes just described, we can compare them with the multidimensional constellation schemes proposed by Forney and Wei (cf. [7], Table II, p. 891). The results are gathered in Table III. It appears that similar performance (2D normalized power for a given rate) can be obtained using only 2-dimensional constellations, hence in a simpler way. Moreover, we believe that the 2D signaling schemes have the advantage of allowing a more realistic comparison with the coded modulation systems at fractional spectral efficiencies. We think this could be more suitable than the baseline criterion proposed by Forney and Wei [7].

It should be mentioned however that practical implementation of these systems for transmitting at a fractional rate is made difficult by the asynchronous nature of signaling. If a detection error occurs, there exists a risk of deletion of a bit, or insertion of a spurious one. Specific means should be provided in order to limit the incidence of such an event [14].

## V. CONCLUDING REMARKS

We presented an attractive mapping for square constellations as well as a simple detection algorithm which seems to be interesting for practical implementation of QAM at high-speed data transmission. In particular, it would be of interest to high-capacity digital microwave radio systems. We obtained also upper and lower bounds on the symbol error probability in the presence of band-limited AWGN for 2D generalized cross constellations which include as a special case the conventional cross constellations ($M = 2^m, m$ odd). The ability of 2D cross constellations in transmitting fractional rates was investigated making use of an opportunistic secondary channel. A closed-form expression of the normalized rate (in bits per 2D) as a function of the constellation size was obtained. These signaling schemes are suggested as a reference for performance comparison with coded-modulation systems at fractional normalized rates.

TABLE III. — Comparison between 2D and multidimensional uncoded signaling schemes.

| 2D generalized constellations | | | | | | Multidimensional constellations | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $M$ | $\beta$ | $N$ | $P(C)$ | dB | PAR | $\lvert C_2 \rvert$ | $\beta$ | $N$ | $P(C)$ | dB | PAR |
| 128 | 7.0000 | 2 | 81.75 | 19.12 | 2.07 | 128 | 7.0000 | 2 | 81.75 | 19.09 | 2.07 |
| 144 | 7.1250 | 2 | 91.78 | 19.63 | 1.94 | 144 | 7.1250 | 16 | 87.24 | 19.39 | 2.10 |
| 160 | 7.2500 | 2 | 101.80 | 20.08 | 1.98 | 160 | 7.2500 | 8 | 94.24 | 19.72 | 2.16 |
| 180 | 7.4062 | 2 | 114.71 | 20.60 | 1.97 | 180 | 7.3750 | 16 | 101.80 | 20.06 | 2.25 |
| 192 | 7.5000 | 2 | 122.33 | 20.88 | 1.98 | 192 | 7.5000 | 4 | 112.04 | 20.47 | 2.18 |
| 216 | 7.6875 | 2 | 137.41 | 21.38 | 1.99 | 216 | 7.6250 | 16 | 121.20 | 20.81 | 2.27 |
| 240 | 7.8750 | 2 | 152.93 | 21.85 | 1.95 | 240 | 7.7500 | 8 | 132.12 | 21.19 | 2.31 |
| 256 | 8.0000 | 2 | 162.75 | 22.12 | 1.93 | 256 | 8.0000 | 2 | 162.75 | 22.10 | 2.00 |
| 270 | 8.0547 | 2 | 171.90 | 22.35 | 2.01 | 270 | 7.8750 | 16 | 143.72 | 21.55 | 2.39 |

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