The RANDOM CODED MODULATION: PERFORMANCE AND EUCLIDEAN DISTANCE SPECTRUM EVALUATION

H. Magalhães de Oliveira and G. Battaile,
TELECOM PARIS, Dept. COM,
46, rue Barrault, 75634 PARIS CEDEX 13, France

Bounds on the error probability

This paper is intended to apply Shannon’s random coding argument [1] to coded modulation [2]. By a union bound argument, we show that

\[ P_e(N) \leq \sum_{d_{min}} \frac{N_{d_{min}}(d^2)}{2d^2} \text{erfc}(\sqrt{d^2}) \]

where \( P_e(N) \) is the error probability of an N-dimensional (N-D) constellation with \( M \) equally likely signals over an additive white and Gaussian noise (AWGN) channel with double-sided spectral density \( N_0/2 \) W/Hz, \( a > 0 \), \( d^2 = \frac{\gamma_0}{\gamma} \), \( \gamma \) = Signal-to-noise ratio, \( S \) is the average normalized power per 2-D symbol, and \( N_{d_{min}}(d^2) \) is the average number of points at a squared Euclidean distance \( d^2 \) from a given point in the (coded or uncoded) constellation.

Let now an N-D uncoded constellation result from the Cartesian product of a constituent 2-D constellation (of size \( q \) and minimum distance \( d_{min} \)) with itself \( N \) times. We have \( \binom{N}{d_{min}} \) possible ways to choose the \( d_{min} \) points, of which \( M \) are \( M \) of these \( d_{min} \) points, whose total distance \( R = \sum_{d_{min}} \frac{N_{d_{min}}(d^2)}{2d^2} \text{erfc}(\sqrt{d^2}) \), where \( R \) is the probability of the \( N \)-dimensional constellation.

Let the dimension of the uncoded and random coded constellation be \( M_{d_{min}}^2 = N_{d_{min}}(d^2) \) for \( d_{min} \) and \( P_d^2 = N_{d_{min}}(d^2) \) for \( d_{min} \) respectively. Assuming that the multidimensional random coded constellation has the same distribution as the uncoded multidimensional signal set is as large as \( N_{d_{min}}(d^2) \) and \( N_{d_{min}}(d^2) \) for \( d_{min} \) and \( d_{min} \) respectively. By a union bound argument, we show that

\[ P_e(N) \leq \sum_{d_{min}} \frac{N_{d_{min}}(d^2)}{2d^2} \text{erfc}(\sqrt{d^2}) \]

where \( P_e(N) \) is the average error probability of a random coded modulation with an N-bit signal set and \( \gamma \) is a constant defined by:

\[ N_{d_{min}}(d^2) = \sum_{d_{min}} \frac{N_{d_{min}}(d^2)}{2d^2} \text{erfc}(\sqrt{d^2}) \]

where \( \gamma \) is the proportion of the total distance \( R \) between the \( N \)-dimensional and \( d_{min} \)-dimensional constellations.

Using (1) instead of (2) in the derivation of (3) results in a bound for a particular code (coded constellation) instead of the random code. Hence, an interesting way to compare codes is through their respective partial cutoff rates whose computation involves their (hard to obtain) complete distance spectrum. We have the simpler bounds \( \bar{R}_d(R_0) \leq \bar{R}_d(0) \leq \bar{R}_d(R_{0\text{min}}) \), where \( \bar{R}_d(R) = \sum_{d_{min}} \frac{N_{d_{min}}(d^2)}{2d^2} \text{erfc}(\sqrt{d^2}) \), standing here for the subscript min or mean. Clearly, the previous bounds are useless near and above \( R_0 \).

Asymptotic Behaviour of Very Long Codes

Given an arbitrary \( \epsilon > 0 \), if we pick at random a particular coded (redundant) signal set \( \mathbf{S} \) of length \( N \), then it is almost surely a good one (i.e., \( P_e(N) < \epsilon \), provided that \( R > R_0 \). We define the epsilon-distignuishability as the minimal distance between any two different \( \epsilon \)-independent sets, two different \( \epsilon \)-independent sets are said to be quasi-identical if their normalized squared distance distributions are \( \epsilon \)-independent. For any \( \epsilon > 0 \), almost all codes (coded constellations) are quasi-identical to the average code (random code), provided \( N \) is large enough, and almost all codes become quasi-identical for any \( \epsilon > 0 \) as \( N \) increases without limit. Concluding Remarks

We applied Shannon’s random coding argument to coded modulation. Asymptotically, we found that the normalized squared Euclidean distance exhibits a hardening phenomenon. This confirms that the minimal distance is not significant for long codes [5] and may be related to our recent finding that lattices exist which achieve the channel capacity [6]. We also showed that virtually all large coded signal sets are good provided the error rate does not exceed a critical value.

References