Applications of Non-Orthogonal Filter Banks to Signal and Image Analysis

Luciana R. Soares, *Student Member, IEEE*, H. M. de Oliveira, *Member, IEEE*, and Renato J. S. Cintra

***Abstract*—A non-orthogonal wavelet-based multiresolution analysis was already provided by scaling and wavelet filters derived from Gegenbauer polynomials. Allowing for odd *n* (the polynomial order) and α value (a polynomial parameter) within the orthogonality range of such polynomials, scaling and wavelet functions are generated by frequency selective FIR filters. These filters have compact support and generalized linear phase. Special cases of such filter banks include Haar, Legendre, and Chebyshev wavelets. As an improvement, it has been achieved that for specific α values it is possible to reach a filter with flat magnitude frequency response. We obtain a unique closed expression for α values for every *n* odd value. The main advantages in favor of Gegenbauer filters are their smaller computational effort and a constant group delay, as they are symmetric filters. Potential applications of such wavelets include fault analysis in transmission lines of power systems and image processing.**

***Index Terms*—Chebyshev wavelets, discrete-time filters, filter banks, Gegenbauer polynomials, Gegenbauer wavelets, image analysis, Legendre wavelets, multiresolution analysis, signal analysis, wavelet transform.**

I. Introduction

Recently, new non-orthogonal wavelet families were introduced [1]-[3] based on the idea of linking 2nd-order ordinary differential equations with the transfer function of multiresolution analysis filters. These families are related to Legendre, Chebyshev and Gegenbauer polynomials, the latter one being a broad class that generalizes the earliest two.

Filter banks based on the wavelet-based multiresolution analysis (WMRA) have already been stated as a powerful tool for signal and image analysis [4]-[5]. The choice of an appropriate mother wavelet is the earliest step when performing such an analysis. Orthogonal or biorthogonal filter banks are always chosen for analyzing signals and images due to the perfect reconstruction property [4]. Nevertheless, this paper shows that interesting results can also be derived from non-orthogonal WMRA when analyzing some kind of signals and images.

Two Gegenbauer WMRA are presented. The first one is named selective Gegenbauer WMRA and performs a lossy analysis. The second one is the flat Gegenbauer WMRA and makes a redundant analysis. Both Gegenbauer WMRA are derived from Gegenbauer polynomials, being connected to the multiresolution analysis and wavelet filters.

The selective Gegenbauer WMRA filter banks have compact support, and are furthermore derived from FIR generalized linear phase filters, i.e., constant group delay filters. In this kind of analysis, *n* (the polynomial order) and α (a polynomial parameter) control the selectivity of both Gegenbauer low-pass and high-pass filters. Consequently, these parameters must carefully be selected. This approach has already been suggested for fault analysis in transmission lines [6].

As an improvement on filter banks derived from Gegenbauer polynomials, this paper shows how to reach Gegenbauer filter banks with flat magnitude frequency response by setting particular α values. In that case, the resulting scaling and wavelet filters are FIR linear phase filters.

The selective Gegenbauer WMRA is first revisited and then the flat Gegenbauer WMRA is introduced. Both Gegenbauer filter banks can be useful for signal and image processing, thereby being an alternative to any orthogonal filter bank, especially when a fast computing is required or when group delay plays an important role, such as in real-time image processing.

II. The Selective Gegenbauer Multiresolution Analysis

Gegenbauer polynomials are solution of the differential equation, *n* integer:

$$(1-z^2)\frac{d^2y}{dz^2}-(2\alpha+1)z\frac{dy}{dz}+n(n+2\alpha)y=0. \qquad (1)$$

The *n*th-order orthogonal Gegenbauer polynomial $C_n^{(\alpha)}(z)$ can be found, for $n>2$, $|z|\le 1$ and $\alpha>-1/2$, by the recurrence relation [7]:

$$C_n^{(\alpha)}(z)=\frac{1}{n}\cdot\left[2\cdot(\alpha+n-1)\cdot z\cdot C_{n-1}^{(\alpha)}(z)-(2\cdot\alpha+n-2)\cdot C_{n-2}^{(\alpha)}(z)\right], \qquad (2)$$

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L. R. Soares and H. M. de Oliveira are with the Signal Processing Group, Department of Electronics and Systems, Federal University of Pernambuco – UFPE, C.P. 7800, 50711-970, Recife, PE, Brazil (e-mail: lusoares@ufpe.br, hmo@ufpe.br). R. J. de Sobral Cintra is with the Department of Statistics of the same university (e-mail: rjdsc@de.ufpe.br)

with $C_1^{(\alpha)}(z) = 2 \cdot \alpha \cdot z$ and $C_2^{(\alpha)}(z) = 2 \cdot \alpha \cdot (\alpha+1) \cdot z^2 - \alpha$. Additionally, $C_n^{(\alpha)}(-z) = (-1)^n \cdot C_n^{(\alpha)}(z)$ holds.

It was shown [3] that the variable change $z = \cos(\omega/2)$, $n = v$, v odd, under the constraint α strictly positive, generates a low-pass frequency selective FIR filter, in such a way that its impulse response converges to a scaling function of a MRA. In spite of the fact that Gegenbauer polynomial holds orthogonality for $\alpha > -1/2$, it has not a low-pass behavior within the interval $-1/2 < \alpha < 0$ [3]. Furthermore, 1st-order Chebyshev polynomials (any vth-order Gegenbauer polynomial with $\alpha = 0$) do not generate scaling functions of a MRA [2].

The Gegenbauer scaling filter was defined by [3]:

$$H_v(\omega) = \frac{C_v^{(\alpha)}(\cos(\omega/2))}{C_v^{(\alpha)}(1)} \cdot e^{-j\frac{\omega v}{2}}, \qquad (3)$$

where $C_v^{(\alpha)}(.)$ is a v^{th}-order Gegenbauer polynomial.

After some trigonometric handling, it can be shown that the Gegenbauer filter coefficients are given by [3]:

$$\frac{h_k^v}{\sqrt{2}} = \frac{1}{C_v^{(\alpha)}(1)} \frac{\Gamma(\alpha+k)\Gamma(\alpha+v-k)}{k!(v-k)!\Gamma^2(\alpha)}, \qquad (4)$$

where $k = 0,1,\ldots,v$, $\Gamma(.)$ is the standard Euler gamma function, $\alpha \neq 0$ and $0 < \theta \leq \pi$.

The number of zero crossings within the interval $0 < \omega \leq 2\pi$ depends on the degree (v) of the Gegenbauer polynomial. In the Z-plane, all zeroes are located on the unit circle. For a preset v, the parameter α also rules the main lobe width as well as the stop-band attenuation of the Gegenbauer scaling filters. Illustrative examples of such scaling filters are shown in the Fig. 1. It can be seen that these scaling filters are low-pass selective FIR filters.

Despite the fact of Gegenbauer scaling filters always hold the *necessary condition* for orthogonality, $v = 1$ is the sole Gegenbauer filter order that generates an orthogonal MRA. Actually, the Gegenbauer filter ($v = 1$, any α) collapses into the Haar filter, which is the only orthogonal symmetrical filter [8]-[9].

A Gegenbauer wavelet filter of a non-orthogonal WMRA can be implemented through a π-shift on the frequency response of the Gegenbauer scaling function. In discrete time domain, such a frequency-shift corresponds to a circular shift in the low-pass filter coefficients, i.e., $g_k = (-1)^k h_{1-k}$, $k = 1,2,\ldots,v$, where g_k are the wavelet filter coefficients.

Defining a non-orthogonal filter bank based on low-pass and high-pass Gegenbauer filters, the scaling and wavelet waveforms can be derived by the cascate algorithm [9]. Fig. 2 shows a few Gegenbauer scaling and wavelet functions.

Since $\alpha \in \mathbf{R}$, $\alpha > 0$, the Gegenbauer WMRA family has uncountable Gegenbauer scaling and wavelet functions for any v (positive) odd. Special cases of the selective Gegenbauer WMRA family are Haar ($v = 1$, any α), Legendre (any v, $\alpha = 0.5$) [1], and 2nd-order Chebyshev family (any v, $\alpha = 1.0$) [2].

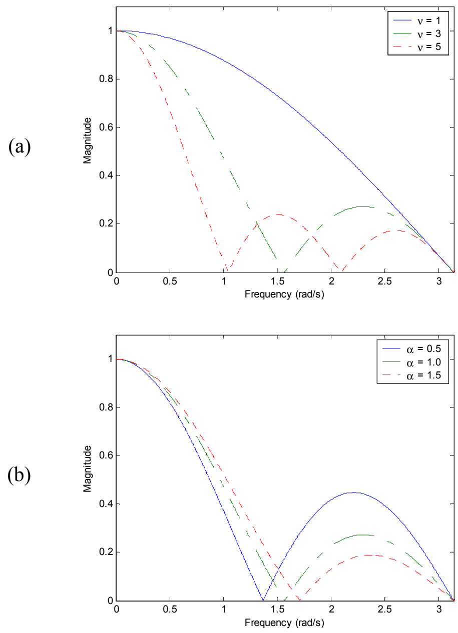
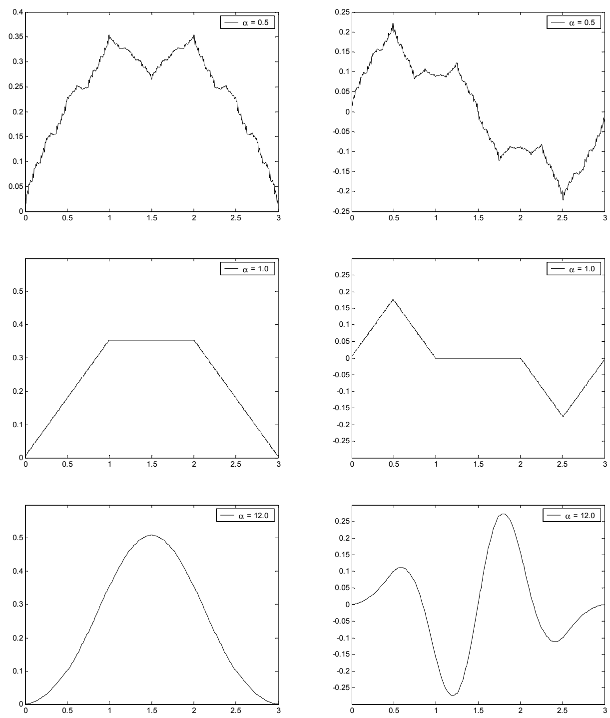


Fig. 1. Magnitude frequency response of Gegenbauer scaling filters: (a) $\alpha = 1$ and $v = 1, 3, 5$; (b) $\alpha = 0.5, 1.0, 1.5$ and $v = 3$.



(a) (b)

Fig. 2. Gegenbauer waveforms derived from 4-coefficient filters after four iterations using $v = 3$ and $\alpha = 0.5, 1.0, 12.0$: (a) scaling functions; (b) wavelet functions.

Gegenbauer WMRA family presents scaling filters with even symmetry and wavelet filters with odd symmetry. They are therefore, type II (even length and even symmetry) and type IV (even length and odd symmetry) FIR generalized linear phase filters [10], respectively.

III. Gegenbauer Filter Banks with Flat Frequency Response

As the variable change $z = \cos(\omega/2)$ only imposes that $\alpha \neq 0$ [7] and (4) has been derived regardless the α value influence, the use of negative α values is allowed.

The gamma function diverges for some negative values. However, instead of searching these α values, we explore those ones that yield a flat magnitude frequency response of the scaling filter at the origin, $\omega = 0$.

Thus, supposing that a Gegenbauer scaling filter has flat and unitary-magnitude frequency response in the vicinity of $\omega = 0$, $d\left[C_v^{(\alpha)}(\cos(\omega/2))\right]/d\left[\cos(\omega/2)\right] = 0$ and $C_v^{(\alpha)}(1) = 1$, we derive $\alpha = -v/2$ for any odd Gegenbauer polynomial order (v). A similar result was obtained by Soltis [11] when proposing a wavelet design in the continuous-time domain. None filter coefficients were provided.

The flat Gegenbauer frequency response and their filter coefficients are also given, respectively, by (3) and (4).

For a preset v, the parameter α is already defined ($\alpha = -v/2$) and it rules the cutoff frequency of the flat Gegenbauer scaling filters. Illustrative examples of such scaling filters are shown in the Fig. 3. It can be seen that these scaling filters have flat low-pass response, are naturally normalized, having unitary-magnitude at the origin, and are linear phase filters [10].

The number of zero crossings within the interval $0 < \omega \leq 2\pi$ is unitary for any odd degree (v) of the Gegenbauer polynomial. Moreover, in the Z-plane there is just one zero at $z = -1$ and the others zeroes are located on conjugated positions.

As in the case $\alpha > 0$, the flat Gegenbauer scaling filters always hold the *necessary condition* for orthogonality. However, just for $v = 1$ there is an orthogonal MRA, reaching again the Haar filter.

Fig. 4 shows the results of the iterative procedure for the convergence test [9] of the impulse responses of 3rd and 5th Gegenbauer order filters. It can be observed that there is convergence as the iteration number grows. So, these impulse responses can be named as scaling functions of the flat Gegenbauer MRA.

Since these impulse responses converge to scaling functions of the MRA, a non-orthogonal filter bank based on low-pass and high-pass flat Gegenbauer filters can be created and the scaling and wavelet waveforms derived. Fig. 5 shows a few Gegenbauer scaling and wavelet functions for some v orders and $\alpha = -v/2$.

The flat Gegenbauer WMRA family presents also scaling filters with even symmetry and wavelet filters with odd symmetry. However, in contrast to the selective WMRA, composed by generalized linear phase filters, the filter banks of the flat WMRA have strictly positive magnitude (Fig. 3). As a consequence, they are FIR linear phase filters [10].

(a)

(b)

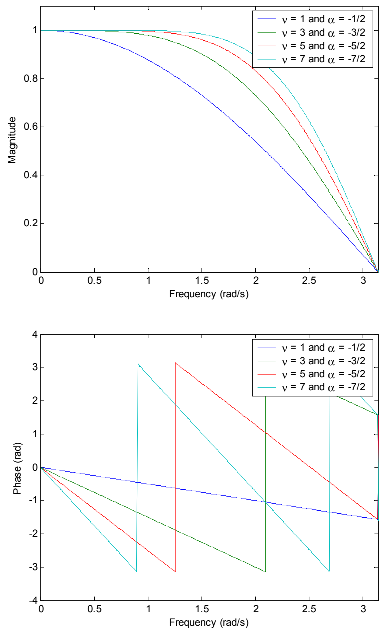


Fig. 3. Frequency response of Gegenbauer low-pass filters for $\alpha = -v/2$ and $v = 1, 3, 5, 7$: (a) magnitude response; (b) phase response.

(a)

(b)

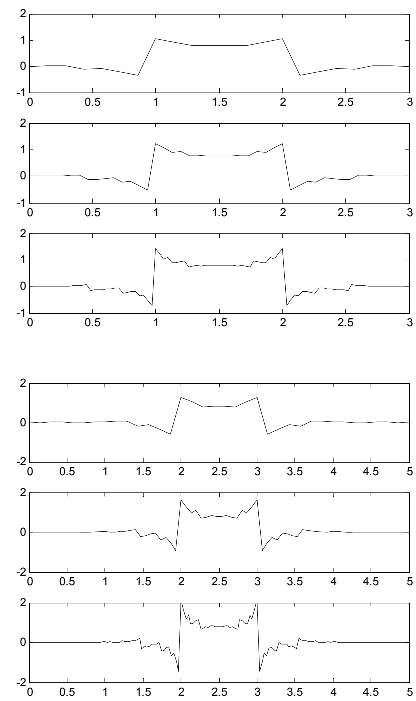
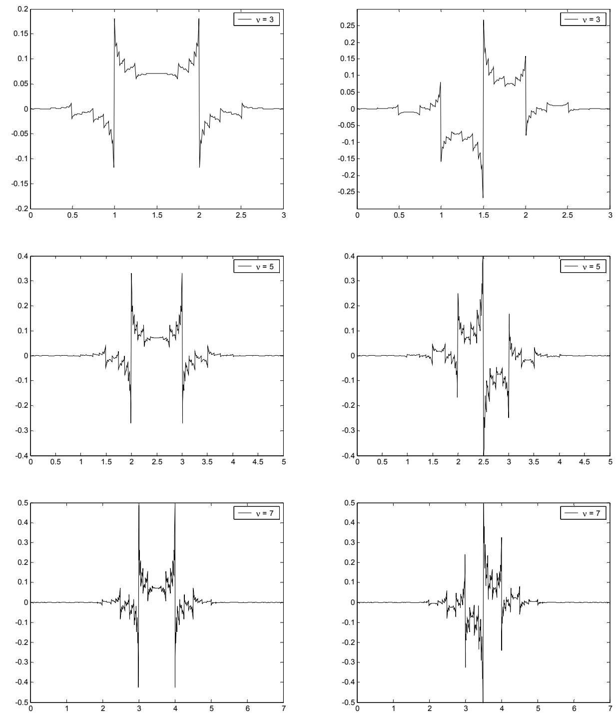


Fig. 4. Scaling Gegenbauer waveforms derived from 4-coefficient and 6-coefficient filters after four iterations for $\alpha = -v/2$: (a) $v = 3$, (b) $v = 5$. The convergence can be observed as the iteration number grows.



(a) (b)

Fig. 5. Gegenbauer waveforms derived from 4-coefficient, 6-coefficient and 8-coefficient filters after four iterations under the constraints $v = 3, 5, 7$ and $\alpha = -v/2$: (a) scaling functions; (b) wavelet functions.

IV. On Selecting The Gegenbauer Filter Bank

A. Frequency Bands

Tables I, II and III show three decomposition levels for a signal sampled at 128 samples/cycle, when using an orthogonal WMRA and some Gegenbauer WMRA filter banks. The cut-off frequency was obtained at –3 dB.

Let Geg$v$aα denote a selective Gegenbauer filter bank with parameters v and α, e.g., Geg3a1 denotes a Gegenbauer (Chebyshev) filter bank with $v = 3$ and $\alpha = 1$. As v and α values drive the selectivity of such Gegenbauer filters, these parameters should carefully be selected.

For the flat Gegenbauer filter bank (with parameters v and $\alpha = -v/2$), consider Geg$v$a**r**, **r** characterizing the redundant analysis because of the superposing frequency bands on both scaling and wavelet filters.

The choice of the kind of Gegenbauer filter bank depends upon what it is intended for.

TABLE I

SIGNAL LENGTHS AND FREQUENCY BANDS AT THREE DECOMPOSITION LEVELS OF AN ORTHOGONAL WAVELET MULTIRESOLUTION ANALYSIS

Level (j)	**Signal length** (samples/cycle)	**Scaling Filter** (Hz)	**Wavelet Filter** (Hz)
1	64	0 – 1920	1920 – 3840
2	32	0 – 960	960 – 1920
3	16	0 – 480	480 – 960

TABLE II

FREQUENCY BANDS AT THREE DECOMPOSITION LEVELS FOR 3RD-ORDER LEGENDRE AND CHEBYSHEV WAVELET MULTIRESOLUTION ANALYSIS

Level	**Scaling Filter** (Hz)		**Wavelet Filter** (Hz)	
(j)	**Geg3a0.5**	**Geg3a1**	**Geg3a0.5**	**Geg3a1**
1	0 – 795	0 – 877	3045 – 3840	2963 – 3840
2	0 – 398	0 – 439	1522 – 1920	1481 – 1920
3	0 – 199	0 – 219	761 – 960	741 – 960

TABLE III

FREQUENCY BANDS AT THREE DECOMPOSITION LEVELS FOR A FEW 3RD-ORDER GEGENBAUER WAVELET MULTIRESOLUTION ANALYSIS

Level	**Scaling Filter** (Hz)		**Wavelet Filter** (Hz)	
(j)	**Geg3a12**	**Geg3ar**	**Geg3a12**	**Geg3ar**
1	0 – 1110	0 – 2513	2730 – 3840	1327 – 3840
2	0 – 555	0 – 1256	1365 – 1920	664 – 1920
3	0 – 278	0 – 628	682 – 960	332 – 960

B. Computational Effort and Group Delay

Due to its symmetry, both Gegenbauer WMRA families present a half of the computational effort compared to any same length asymmetrical filter bank. In particular, setting $\alpha = 1$, the vth-order Gegenbauer (Chebyshev) filter has identical coefficients, requiring lesser computational effort than any asymmetrical filter.

Additionally, every single one filter has linear phase and constant group delay, given by $v/2$, which means that *no different delay is introduced at different frequencies* of the analyzed signal.

V. Applications of Gegenbauer Filter Banks

Both Gegenbauer WMRA families can be applied for signal and image analysis. The selective filter banks ($\alpha > 0$), $v \neq 1$, perform lossy analysis and the flat filter banks ($\alpha = -v/2$), $v \neq 1$, implement a redundant analysis, when compared to an orthogonal filter bank.

A. Signal Analysis Applications

The selective Gegenbauer WMRA has already been suggested for fault analysis in transmission lines [6]. In this paper, the high-frequency components are used for fault detection and the low-frequency components for fault location. The apparent impedance approach was used to estimate the fault distance from the monitoring terminal, throughout fundamental components of approximated versions of the voltage and current signals. Possible fundamental component attenuation due to the filter selectivity is not critical for distance searching.

Fig. 6 shows one voltage signal derived from a simulated three-phase fault on a transmission line and their detailed versions after one decomposition level using the following four-coefficients filter banks: an orthogonal one (Daubechies: Daub4), and both Gegenbauer filter banks (Geg3a1 and Geg3ar). Fig. 7 shows the same voltage signal and their approximated versions after three decomposition levels using the same filter banks.

The flat Gegenbauer WMRA seems useful for these fault detection algorithm. However, the fault detection threshold might be selected carefully. The third-level approximated versions derived from the flat Gegenbauer WMRA may not show soft oscillations after the fault occurrence due to its broad low-pass frequency response.

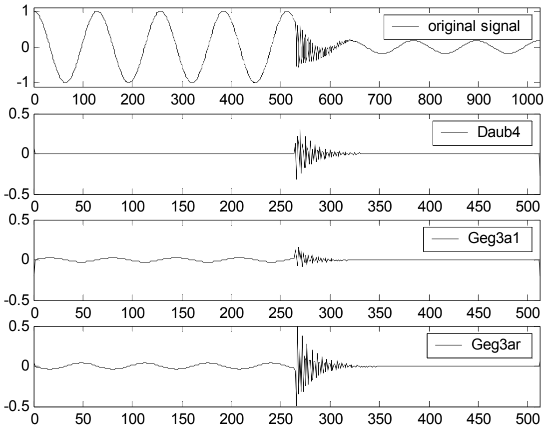


Fig. 6. A voltage signal and their detailed versions after one decomposition level using the following filter banks: Daub4, Geg3a1, and Geg3ar.

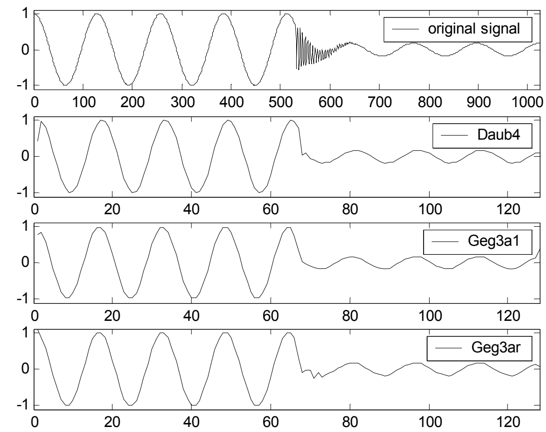


Fig. 7. A voltage signal and their approximated versions after three decomposition levels using the following filter banks: Daub4, Geg3a1, and Geg3ar.

B. Image Processing Applications

One feature of linear-phase filters is their tendency to preserve the shape of the signal component in the passband region of the filter. A nonlinear phase distorts the proper registration of different frequency components that make up the lines and scratches on images. This may distort the signal shape in various ways, including blurring [12].

The linear phase property appears to be more important on image processing than on signal processing. As the proposed Gegenbauer filter banks have linear phase, they seems also useful for image processing.

The two-dimensional WMRA can be conceived as a one-dimensional WMRA along the x and y axes of an image. This two-dimensional analysis leads to a decomposition of approximation coefficients at level j in four components at level $j+1$: the approximation, and the details in three orientations (horizontal, vertical, and diagonal). Thus, one scaling and three wavelet functions are used for such analysis and they are based on tensorial product derived from one-dimensional ones [4], [13].

Image compression is one of the most useful applications of image processing using the WMRA [4]-[5]. Possible applications of such filter banks in the power system area may include image transmission of monitored substations.

Fig. 8 shows an image (460 × 460 pixels) from a 69/13.8kV substation area supervised by a monitoring system, its compressed image after two decomposition levels using Geg5ar filter banks and the difference image between them.

The compression algorithm was used supposing a soft algorithm for the wavelet coefficients elimination, '*sorh = s*', and a global threshold for every decomposition level, "*thr = 20*' [13]. It results in a compressed image consisting of about 76.7% zeros while retaining 98.2% the energy of the original image.

Curiously, the difference image may perform a sketch of the original image, which might not be attained by orthogonal WMRA.

Flat Gegenbauer filter banks may be useful for signal enhancement, since the reconstructed image may evidence high frequencies components of the original image, which can be seen in Fig. 8.

VI. Conclusions

The selective Gegenbauer WMRA was revisited with the purpose of show that the flat Gegenbauer WMRA can also be obtained by identical filter coefficients expression.

The scaling and wavelet filters of the flat Gegenbauer WMRA can be attained throughout odd (v) order polynomial parameters and a unique α value: $\alpha = -v/2$. They have compact support and are FIR linear phase filters. Additionally, they have flat magnitude frequency response.

As Gegenbauer WMRA families are composed by symmetrical filters, they introduce the same delay at different frequencies, given by $v/2$, and require lesser computational effort than any asymmetrical filter bank.

The choice of the kind of Gegenbauer filter bank depends upon what it is intended for. These initial findings indicate that Gegenbauer filter banks can also be a promising filter bank for image analysis.

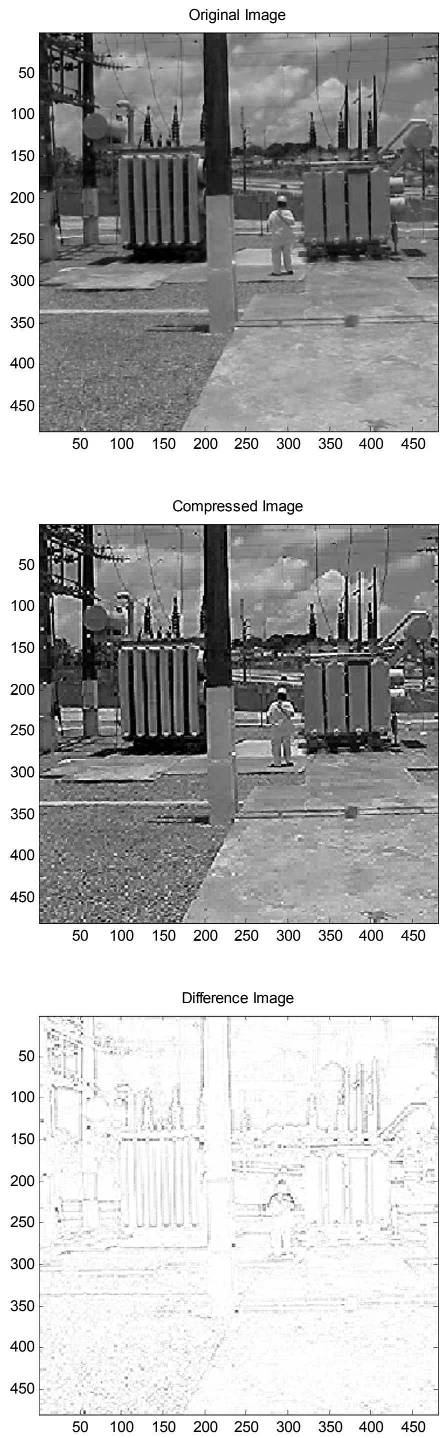


Fig. 8. Image from a 69/13.8kV substation area, its compressed image after two decomposition levels using Geg5ar filter banks and the difference image between them.

VII. References

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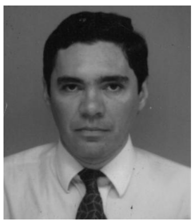
VIII. Biographies

Luciana R. Soares (S'2004) was born in Rio de Janeiro, Brazil, on May 30, 1973. She received the B.Sc and M.Sc degrees in electrical engineering from Universidade Federal de Pernambuco (UFPE), Recife, Brazil, in 1997 and 2001. Currently, she is a D.Sc. candidate at the Department of Electronics and Systems at same university (DES-UFPE).

Her special fields of interest include power quality monitoring and analysis, electromagnetic compatibility, fault analysis, wavelets, and applications of wavelet transforms on power system.



Hélio M. de Oliveira (M'1994) was born in Arcoverde, Pernambuco, Brazil, on May 1, 1959. He received the B.Sc. and M.S.E.E. degrees in electrical engineering from Universidade Federal de Pernambuco (UFPE), Recife, Brazil, in 1980 and 1983. Since 1983 he has been with the Department of Electronics and Systems (DES-UFPE). In 1992 he earned the *Docteur* degree from *École Nationale Supérieure des Télécommunications*, Paris.

Dr. de Oliveira was appointed as honored professor by about 20 undergraduate classes and authored the book *Análise de Sinais para Engenheiros: Wavelets* (Signal Analysis for Engineering: Wavelets). His research topics include communication theory, information theory and signal analysis. He is member of IEEE and SBrT (Brazilian Telecommunication Society).

Renato J. de Sobral Cintra was born in Recife, Pernambuco, Brazil. He received the B.Sc., M.Sc. and D.Sc. degrees all in electrical engineering from Universidade Federal de Pernambuco (UFPE), Recife, Brazil. During 2003-2004, he was research assistant at the University of Calgary, Canada. Since 2005 he has been with the Department of Statistics at UFPE. His fields of interest include non-conventional transform techniques, digital signal processing, and biological signal processing.