Homophonic Sequence Substitution

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Abstract — Homophonic sequence substitution is the name given in this paper to the technique which consists of substituting one-to-one a given finite (or semi-infinite) sequence of symbols by another finite (or semi-infinite) sequence over the same alphabet but having a higher entropy rate. It is proved that by sequentially encoding the output of a discrete stationary and ergodic source with binary lossless codes makes the entropy rate of the resulting encoded binary sequence asymptotically approach the value 1, therefore performing optimum homophonic sequence substitution. The cleartext redundancy after k consecutive encodings is 1 - H_k (S) bits per binary digit, where H_k (S) is the entropy rate of the binary sequence resulting after the k-th encoding. A Markov source model is presented to describe the binary encoded sequences and to compute their entropy rate.

1. Introduction

Source coding is a technique whose aim is to represent the output of an information source with as few code digits per source symbol as possible. In this paper we will consider only lossless source coding in which case it is possible to reconstruct exactly the source output from its encoded representation. We will concentrate our attention on binary coding both for its practical importance and because the generalizations to higher order alphabets are immediate. We will consider the problem of removing the redundancy of a message sequence with an alternative, and perhaps complementary, approach to that in [3]. The distinguishing feature of our approach is that we neither resort to intentional plaintext expansion, as in conventional (symbol) homophonic substitution, nor to coding extensions of the original source, as suggested by Shannon's lossless source coding theorem [1, p.69]. In Section 2 we present basic notions of source coding and briefly review the main properties of uniquely decodable codes. In Section 3 we define the Markov source associated with a rooted tree with probabilities [5] and consider encoding the output of such a source with a Huffman code. In Section 4 we introduce alternate Huffman codes and give an example. Following [3] we will call a sequence of D-ary random variables completely random if each of its digits is statistically independent of the preceding digits and is equally likely to take on any of the D possible values. Finally, in Section 5 we show how to perform homophonic sequence substitution and prove that a cascade consisting of Markov sources encoded by lossless codes produces in the limit a completely random sequence.

The decoding operation is very simple and consists of applying the encoded binary sequence through a cascade of k look-up tables (corresponding to the number k of iterations), where the i-th, 1 ≤ i ≤ k, look-up table in the cascade is a decoder for the (k + 1 - i)th code. The nice aspect in this approach is that the resulting implementation complexity remains small, because it grows linearly with the number of encodings, and there is no cleartext expansion caused by the iterations. Of course the binary sequence after the k-th encoding will still have a redundancy (measured in bits) which is equal to 1 - H_k (S) bits per binary digit, where H_k (S) is the entropy rate of the binary sequence after the k-th encoding.

2. Source Coding Fundamentals

Let U_1, U_2, ..., denote the output sequence of symbols of a discrete information source. This source is said to be stationary if, for every positive integer L and every sequence u_1, u_2, ..., u_L of letters from the source alphabet we have P(U_1 = u_1, ..., U_L = u_L) = P(U_{1+1} = u_{1+1}, ..., U_{L+L} = u_{L+L}), for all L ≥ 0. A stationary source is said to be ergodic if the number of times that the sequence u_1, u_2, ..., u_L occurs within the source output sequence U_1, U_2, ..., U_{N+L-1} of length N = L - 1, when divided by N, equals P(U_1 = u_1, U_2 = u_2, ..., U_L = u_L) with probability 1 as N → ∞ [4]. In the sequel we will consider only discrete stationary and ergodic sources (DSES) since they are general enough to model any real information source. The source codes employed in lossless source coding are called uniquely decodable codes [2, p.48]. A sufficient condition for the unique decodability of a concatenation of codewords is that the encoding be prefix-free, i.e., that no codeword be the first part (prefix) of another codeword. This prefix-free condition is equivalent to the condition that a decoder be able to immediately recognize the end of a codeword without need to read the beginning of the next codeword. Codes with this property are called instantaneous codes [2, p.50]. A uniquely decodable code is further said to be a compact code [2, p.65] whenever its average codeword length is equal to or less than the average codeword length of all other uniquely decodable codes for the same source and the same code alphabet.

Shannon's lossless source coding theorem [1, p.69] implicitly suggests that the only way for reducing redundancy in a message to be transmitted or stored is by performing data compression. As the cryptographic community very well knows that is not necessarily the case however,

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as exemplified by homophonic substitution. **Homophonic substitution** is a cryptographic technique for reducing the redundancy of a message to be encrypted at the cost of plaintext expansion. This definition concerns homophonic symbol substitution [5], however iterative source coding can be seen as a form of homophonic substitution (homophonic sequence substitution) and not necessarily leads to cleartext expansion.

3. **Rooted Trees and Markov Sources**

Very often we are interested in determining the probability of single binary digits, or pairs of binary digits, etc., produced by a source code driven by a source. It turns out that the computation of these probabilities, directly from the code rooted tree with probabilities [5], is possible but becomes very complicated as the order of the statistics considered increases. We found a neater way for calculating these probabilities by defining a representation of the code rooted tree with probabilities by a Markov source. We define the Markov source whose states correspond one-to-one to the nodes of the code tree, whose branches are labeled with the same binary numbers as those in the corresponding branches of the code tree and each state transition probability is given by the conditional probability of emitting a 0 (or a 1) given the current state (or node in the code tree). A return to state $S_i$ occurs whenever the last digit transmitted is the last digit of a codeword. The example below is provided to illustrate the above description of a Markov source and employs a Huffman code.

**Example 1**

Let $S$ denote a discrete source with a four symbol alphabet whose probabilities are $A$, $B$, $C$, and $D$, respectively. We show in Figure 1 the Huffman tree and in Figure 2 the corresponding Markov source for the given discrete source.

4. **Alternate Binary Huffman Codes**

As far as source specific codes for source coding are concerned Huffman codes are compact in the sense that a Huffman code for a specific DSES has an average codeword length equal to or less than the average codeword length among all instantaneous codes for that source [2, p.77] with the same code alphabet. We notice the well known fact that for a given DSES, in general, we can construct more than one Huffman code, but that all such codes have the same average codeword length.

In the construction of a binary Huffman code, or equivalently, a binary Huffman tree, whenever two subtrees stem out from a node a decision has to be made as to which subtree we should label with a 0 and to which subtree we should label with a 1. Whenever that decision is arbitrary the resulting Huffman code is called an arbitrary Huffman code [6]. Whenever the subtree of higher total probability is always labeled with a 0, the resulting code is called a 0-heavy Huffman code. We introduce next a third case of interest that we call alternate Huffman coding. Starting from the root, whenever two subtrees stemming out from the same node have identical probabilities we arbitrarily label one of them with a 0 and the other with a 1. At the first node whose two subtrees stemming out have different probabilities, we label with a 0 the subtree of higher probability and keep a record of that fact. At the next node whose two subtrees stemming out have different probabilities we label with a 1 the subtree of higher probability. This procedure is applied over and over until the tree is traversed. Summarizing, this subtree labeling rule keeps a record of which label was given to the subtree of higher probability at the last node visited whose associated subtrees had different probabilities and alternates that labeling for the next node whose associated subtrees have different probabilities. We illustrate with a simple example the usefulness of alternate Huffman coding.

**Example 2**

Consider a discrete memoryless source whose alphabet has three symbols occurring with probabilities $A$, $B$, and $C$, respectively.

<table>
<thead>
<tr>
<th>Probability</th>
<th>Alternate code</th>
<th>0-heavy code</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$B$</td>
<td>10</td>
<td>00</td>
</tr>
<tr>
<td>$C$</td>
<td>11</td>
<td>01</td>
</tr>
</tbody>
</table>

The entropy per binary digit of the associated Markov source model is identical for both codes and its value is .974. The table below presents first order and second order statistics for both codes. By computing for each code the the absolute value of the difference between each one of the statistics in the table and the corresponding value for a completely random source, and then adding the results
we see that the alternate code produces a smaller sum and thus its digits are more random-looking than those produced by the 0-heavy code. This seems to be a property true in general for alternate codes.

<table>
<thead>
<tr>
<th></th>
<th>Alternate code</th>
<th>0-heavy code</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(0)</td>
<td>4578</td>
<td>90575</td>
</tr>
<tr>
<td>P(00)</td>
<td>1875</td>
<td>30</td>
</tr>
<tr>
<td>P(01)</td>
<td>28125</td>
<td>24375</td>
</tr>
<tr>
<td>P(10)</td>
<td>28125</td>
<td>24375</td>
</tr>
<tr>
<td>P(11)</td>
<td>25</td>
<td>1625</td>
</tr>
</tbody>
</table>

**Definition:** A uniquely decodable code is *optimum* if it is both compact and its symbol statistics is the closest to that of a completely random sequence among all compact codes for that source.

5. **Iterative Procedure**

Let $S$ denote a DSES encoded using a compact binary prefix-free code. We chose to use an alternate Huffman code $C_1$ with average codeword length $L_1$ for that purpose. The iterative procedure for performing homomorphic sequence substitution consists of parsing a concatenation of codewords of $C_1$ in blocks of $r$ digits forming a source $S_1$ with $2^r$ symbols. $S_1$ is then encoded with an alternate binary Huffman code $C_2$ with average codeword length $L_2$. A concatenation of codewords of $C_2$ is then parsed in blocks of $r$ digits forming a source $S_2$ with $2^r$ symbols. $S_2$ is then encoded . . . etc. As we prove in Theorem (1), at each new step the entropy of the resulting binary sequence is increased, if not the block size in the parsing is increased to $r + 1$ and the procedure continues. A stopping rule will specify for a given application that, starting with $r = 2$, the number of steps $k$ is given by the smallest $k$ for which $1 - H_k(S) < e$, where $e \ll 1$ is a small positive quantity. Our proof is more general than needed for the iterative procedure for it employs a general lossless code at no extra increase in difficulty.

**Theorem 1.** Let $S$ denote an entropy $H(S)$ DSES whose output is encoded by a binary lossless code $C_1$ with average codeword length $L_1$. Let us parse a concatenation of codewords of $C_1$ in blocks of $r$ digits forming a source $S_1$ with $2^r$ symbols. We encode $S_1$ with a lossless code $C_2$, etc., and proceed as described above. The entropy rate $H_k(S)$ of the coded sequence at step $k$ is greater than or equal to the entropy rate $H_{k-1}(S)$ of the coded sequence at step $k - 1$, $k = 2, 3, \ldots$. . . .

**Proof:** Let $H_k(S)$ denote the entropy rate of the binary sequence generated by a concatenation of codewords of $C_k$, $k = 1, 2, \ldots$ (starting with $C_1$ driven by $S$). It is well known that $H_1(S) = H(S)/L_1$ and that $H(S_1) = rH_1(S)$. It follows that

$$H_2(S) = H(S_1)/L_2 = rH_1(S)/L_2 \geq H_1(S),$$

where the inequality follows from the observation that $L_2 \leq r$ is an upperbound for the average codeword length of a binary lossless code for a source with $2^r$ symbols. Proceeding with the iterations we obtain at the $k^{th}$ step that $H_k(S) \geq H_{k-1}(S)$, $k = 2, 3, \ldots$. As we proceed with the iterations a step will be reached where the lossless code specified for the source with $r$ symbols will have all codewords with the same length. The resulting binary coded sequence is no longer ergodic and whenever this situation happens we may consider repeating that step however parsing with a block size of at least $r + 1$ symbols and proceed from then on in the same manner or simply to stop. Since the property of increasing entropy of coded sequences is valid for source extensions of any order, it follows that the entropy rate $H_k(S)$ of the $k^{th}$ coded sequence tends in the limit to 1 as the number $k$ of iterations grows.

We notice that whenever this equal codeword length phenomena happens in the iterative procedure the entropy of the coded output sequence is usually quite close to 1.

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**References**


