NEW COMPACTLY SUPPORTED SCALING AND WAVELET FUNCTIONS DERIVED FROM GEGENBAUER POLYNOMIALS

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ABSTRACT

A new family of scaling and wavelet functions is introduced, which is derived from Gegenbauer polynomials. The association of ordinary second order differential equations to multiresolution filters is employed to construct these new functions. These functions, termed ultraspherical harmonic or Gegenbauer scaling and wavelet functions, possess compact support and generalized linear phase. This is an interesting property since, from the computational point of view, only half the number of filter coefficients are required to be computed. By using an alpha factor that is within the orthogonality range of such polynomials, there are generated scaling and wavelet that are frequency selective FIR filters. Potential application of such wavelets includes fault detection in transmission lines of power systems.

1. INTRODUCTION

A recent paper [1] introduced a new wavelet family based on the idea of linking 2nd order ordinary differential equations (wave equations) with the transfer function of multiresolution analysis filters. Gegenbauer polynomials have already been used in low-pass and high-pass filter design [2–5]. In both cases, these polynomials have been associated to frequency response of those filters for window and wavelet design. It was showed [4] that using an alpha factor (α) outside the definition range of orthogonality, it is possible to generate wavelets that are close to Lemarie and discrete gaussian-sinusoid wavelets with frequency response like to half-band filters. This paper shows, in contrast, how to construct wavelet filters that have frequency selective response when considering a within the orthogonality range. The impulse response of Gegenbauer low-pass filters converges to a scaling function of a multiresolution analysis (MRA). It is also possible to generate wavelets by a similar procedure. An example case is presented where Gegenbauer filters are used to analyze fault signals on a power system. Since Daubechies filters are often considered appropriated to analyse that kind of signals [6], the results derived from Gegenbauer filter bank are compared to the ones derived from a Daubechies filter bank.

2. GEGENBAUER POLYNOMIALS

Gegenbauer polynomials, or ultraspherical harmonics polynomials, are solution of the 2nd order differential equation, n integer:

$$(1-z^2)\frac{\mathrm{d}^2y}{\mathrm{d}z}-(2\alpha+1)z\frac{\mathrm{d}y}{\mathrm{d}z}+n(n+2\alpha)y=0. \quad (1)$$

The nth order orthogonal Gegenbauer polynomial $C_n^{(\alpha)}(z)$ is described, for $n>2$, $|z|\leq 1$ and $\alpha>-1/2$, by the recursion relationship [7]:

$$C_n^{(\alpha)}=\frac{1}{n}\left[2(\alpha+n-1)zC_{n-1}^{(\alpha)}(z)-(2\alpha+n-2)C_{n-2}^{(\alpha)}(z)\right], \quad (2)$$

with $C_1^{(\alpha)}(z)=2\alpha z$ and $C_2^{(\alpha)}(z)=2\alpha(\alpha+1)z^2-\alpha$.

Additionally, the following property holds $C_n^{(\alpha)}(-z) = (-1)^nC_n^{(\alpha)}(z)$.

For real z, $|z|\leq 1$, the function $C_n^{(\alpha)}(z)$, n odd, cannot be used as wavelet despite having zero mean value because it presents no decaying-to-zero property, as required [8]. Under later consideration, it is common to adopt the variable change $z=\cos(\theta)$ and deal with the polynomials under the form $C_n(\alpha)(\cos(\theta))$. Figure 1 plots some Gegenbauer polynomials. It can be seen that $C_n(\alpha)(\cos(\theta))$ is π-periodic (n even) without common zeroes; or 2π-periodic (n odd) with a common zero at $\theta=\pi/2$, $|C_n^{(\alpha)}(\cos(\pi/2))|=0$, and has n distinct roots within the interval $0<\theta\leq\pi$ (n even) or $0<\theta\leq 2\pi$ (n odd).

In order to adjust odd degree Gegenbauer polynomials to a suitable frequency response of the low-pass filter, it is necessary to set a null at π. This can be done by assuming θ related to the spectral frequency ω by $\theta=\omega/2$. These scaled versions of Gegenbauer polynomials $|C_n^{(\alpha)}(\cos(\omega/2))|$ present a shape of a low-pass filter. The even degree polynomials do not show low-pass filter behaviour, since $|C_n^{(\alpha)}(\cos(\pi))|\neq 0$. Thus, only odd n are taken into consideration. Constraining α within the polynomial orthogonality range, $\alpha>-1/2$, the number of zeroes in $0<\omega\leq 2\pi$ is driven by the degree of the Gegenbauer polynomial. For a fixed n, it controls the main lobe width and the stop-band attenuation on scaled Gegenbauer polynomials according to:

- $-1/2<\alpha<0$: the main lobe magnitude and width are minor than the side lobes (they are not low-pass!);
- $\alpha=0$: the main lobe width and magnitude are equals to the side lobes (scaled 1st order Chebyshev polynomials);
- $\alpha>0$: the main lobe width and magnitude are greater than the side lobes.

It was shown that the Chebyshev polynomials of 1st kind do not generate scaling functions [1], and consequently do not provide multiresolution analysis. For a fixed n, as α increases, the

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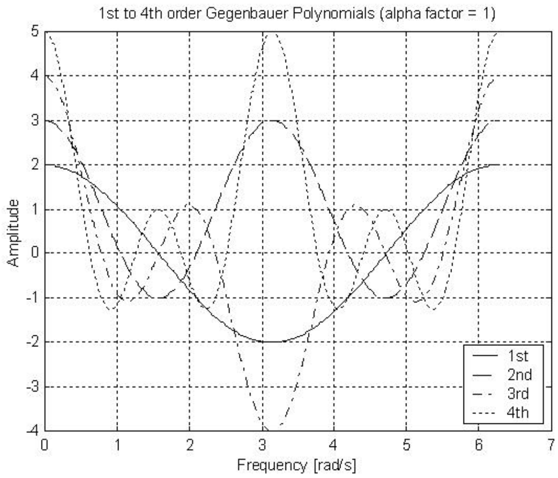
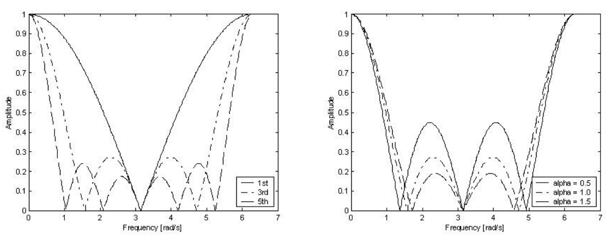


Fig. 1. First to fourth degree Gegenbauer polynomials: $C_n(\alpha)(\cos(\theta))$ for $\alpha = 1$.



(a) (b)

Fig. 2. Magnitude response of Gegenbauer MRA filters: (a) for $\alpha = 1$ and $\nu = 1$, 3 and 5; (b) for $\nu = 3$ and $\alpha = 0.5$, 1.0 and 1.5.

main lobe becomes wider and the slope of attenuation greater. After these remarks, one may restrict α to strictly positive values. Special cases of Gegenbauer polynomials include Legendre polynomials ($\forall n, \alpha = 0.5$) and Chebyshev polynomials of 2nd kind ($\forall n, \alpha = 1.0$).

3. GEGENBAUER MRA

The function $C_n^{(\alpha)}(\cos(\omega/2))$, with $n = 2m + 1 = \nu$, can be used to define the smoothing filter $H(\omega)$ for multiresolution analysis (MRA). An MRA filter must meet appropriate boundary conditions, namely $|H(0)| = 1$ and $|H(\pi)| = 0$ [8]. Let us then assign to $H_\nu(w)$ a magnitude function according to a normalized Gegenbauer polynomials:

$$A_\nu(\omega) = \frac{C_n^{(\alpha)}(\cos(\omega/2))}{C_n^{(\alpha)}(1)}. \quad (3)$$

Illustrative examples of filter transfer functions for some Gegenbauer MRA filters are shown in the Figure 2, for $\nu = 1$, 3 and 5. We can observe the low-pass behaviour of $|A_\nu(\omega)|$, as expected.

3.1. Low-Pass Filter Coefficients

Once defined $A_\nu(\omega)$, the next step is to assign a phase to define $H_\nu(\omega)$ and to compute its filter coefficients h_k, $k \in \mathbb{Z}$. This can be done by applying explicit expressions involving Gegenbauer polynomials and trigonometric functions [7], with $\alpha \neq 0$ and $0 < \theta \leq \pi$:

$$C_n^{(\alpha)}(\cos(\theta)) = \sum_{m=0}^{n} a_m \cos((n-2m)\theta) \quad (4)$$

where $a_m = \frac{\Gamma(\alpha+m)\Gamma(\alpha+n-m)}{m!(n-m)!\Gamma^2(\alpha)}$.

Substituting $\theta = \omega/2$, $n = \nu$ and comparing this expanded equation with that one from type II (even length and even symmetry) finite impulse response (FIR) generalized linear phase filters [9], $H_\nu(\omega)$ can be defined as:

$$H_\nu(\omega) = A_\nu(\omega) e^{-j\frac{\omega\nu}{2}}. \quad (5)$$

And its filter coefficients, associated with Equations 3 to 5, are given by:

$$h_k^\nu = \frac{1}{C_\nu^{(\alpha)}(1)} \left(\frac{a_k + a_{\nu-k}}{2}\right), \quad k = 0, 1, 2, \ldots, \nu. \quad (6)$$

The frequency response of MRA filters is given by $H_\nu(\omega) = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} h_k^\nu e^{-j\omega k}$, so the Gegenbauer MRA filter coefficients are:

$$\frac{h_k^\nu}{\sqrt{2}} = \frac{1}{C_\nu^{(\alpha)}(1)} \frac{\Gamma(\alpha+m)\Gamma(\alpha+n-m)}{m!(n-m)!\Gamma^2(\alpha)}. \quad (7)$$

Since there are $\nu + 1$ nonzero filter coefficients, Gegenbauer low-pass filters have compact support. It is worth to observe that the filter coefficients of Gegenbauer low-pass filters present even symmetry ($h_k^\nu = h_{\nu-k}^\nu$), with centre of symmetry at $\nu/2$. Thus, these new filters are FIR generalized linear phase filters.

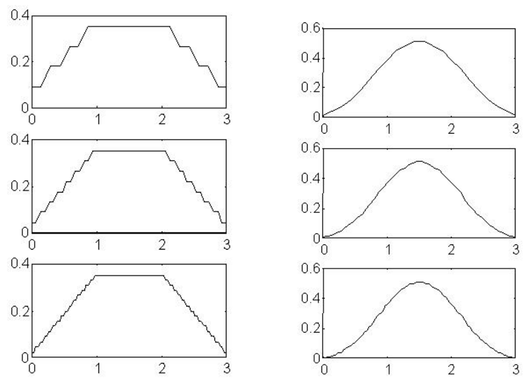
Additionally, the Gegenbauer low-pass filter coefficients satisfy the basic frequency domain MRA conditions and some frequency domain orthogonality conditions [10]:

$$\sum_k h_k = \sqrt{2}, \quad (8)$$

$$\sum_k h_{2k} = \sum_k h_{2k+1} = \frac{1}{\sqrt{2}}. \quad (9)$$

3.2. Scaling Function of an MRA

Although basic MRA conditions are met by Gegenbauer low-pass filter coefficients [10], it is necessary to verify the convergence of the impulse response of these low-pass filters. If L^2-convergence occurs, these impulse responses are scaling functions. There are some methods to verify L^2-convergence [10]. One of them is based on successive approximation of the scaling function from its filter coefficients through an iterative procedure. By using this approach, Figure 3 shows emerging patterns that progressively assume the shape of a few scaling functions of Gegenbauer MRA family. It has been used normalized coefficients to iterative procedure. One important property of scaling functions is orthogonality. This can be investigated verifying the even-shifts (dyadic-shifts) convolution theorem [10]:



(a) (b)

Fig. 3. Gegenbauer scaling function of degree $\nu = 3$ derived after 2, 3 and 4 iterations: (a) $\alpha = 1$; (b) $\alpha = 12$.

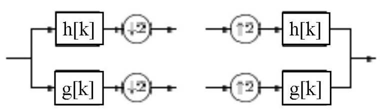


Fig. 4. Non-orthogonal filter bank: h and g corresponds, respectively, to Gegenbauer low-pass and high-pass filters.

$$\sum_k h_k h_{k-2n} = \delta_n, \tag{10}$$

where δ_n is the unit sample sequence. This is a sufficient condition for orthogonality. Although Gegenbauer scaling filters hold necessary conditions for orthogonality, $\nu = 1$ is the only Gegenbauer filter order that generates an orthogonal MRA.

The 1st degree Gegenbauer filter ($\nu = 1$, any α) collapses into the Haar filter, which is the only symmetrical filter that verifies orthogonality [8].

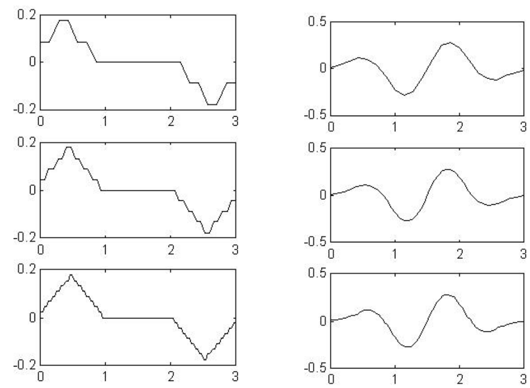
4. GEGENBAUER WAVELETS

In order to define high-pass filters to generate Gegenbauer wavelets, a simple π radians shift on frequency response of the Gegenbauer scaling functions is needed. In discrete time domain, that frequency-shift corresponds to a circular shift of the low-pass filter coefficients. Thus, the wavelet filter coefficients g_k, $k \in \mathbb{Z}$, are given by $g_k^\nu = (-1)^k h_{1-k}^\nu$, $k = 1, 2, \ldots, \nu$.

These wavelets satisfy wavelet properties [8]: zero mean value (sum of its coefficients equals to zero), finite energy (Gegenbauer wavelets have compact support), and admissibility condition.

An non-orthogonal filter bank based on Gegenbauer low-pass and high-pass filters can be generated, Figure 4. The Gegenbauer wavelet shape can be derived by an iterative procedure similar to that one used to generate the scaling function.

Figure 5 shows the wavelet functions corresponding to the scaling functions presented in Figure 3. As with many compact support wavelets, there is no nice analytical formula for describing harmonic ultraspherical wavelets in time domain.



(a) (b)

Fig. 5. Gegenbauer wavelet functions of degree $\nu = 3$ derived after 2, 3 and 4 iterations: (a) $\alpha = 1$; (b) $\alpha = 12$.

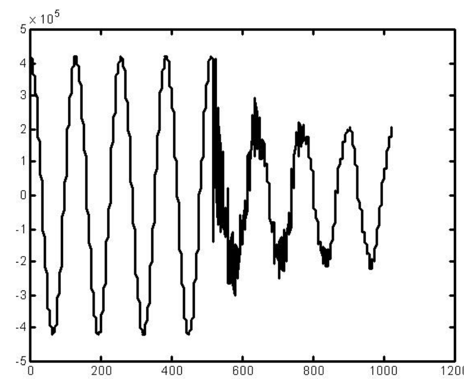


Fig. 6. Single-phase-to-earth fault signal to be analysed.

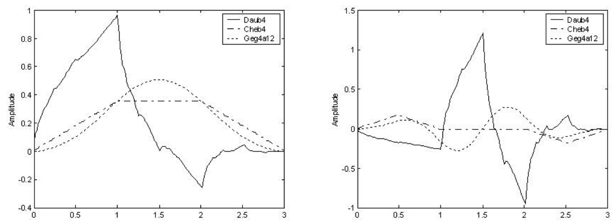
5. EXAMPLE CASE

In the following, a simulated single-phase-to-earth fault signal from a 500kV three-phase power system, presented in Figure 6, is analysed. This signal was analysed with 4-coefficients filters: Daubechies (Daub4), Chebyshev (Cheb4), Gegenbauer with $\alpha = 12$ (Geg4a12). Figures 7 and 8 present the time domain scaling and wavelet functions and its frequency response.

By considering only one stage of filter bank decomposition, the details of that signal are presented in Figure 9. Imposing a threshold to the fault detection, all analysed cases successfully identified the fault condition. Moreover, all detections occur at the same timestamp.

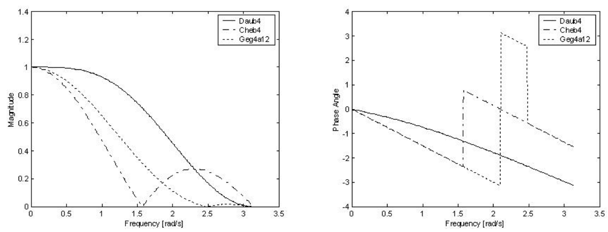
Approximating versions of fault signal is presented in Figure 10 considering three stages of filter banks. Gegenbauer filters present a more soft oscillation after the fault incidence and introduce a delay that is greater than the one offered by Daub4.

These initial findings indicate that Gegenbauer filters can be used to fault analysis. One advantage in favour of Gegenbauer wavelets is the computational effort, since it is derived from symmetric filters.



(a) (b)

Fig. 7. Analysis functions derived from 4-coefficients filters after 5 iterations: (a) scaling; (b) wavelet.



(a) (b)

Fig. 8. Frequency response of the analysing scaling filters: (a) magnitude; (b) phase angle.

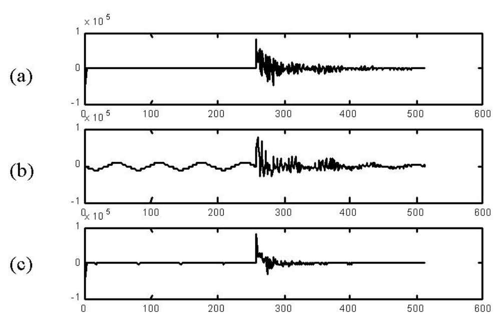


Fig. 9. 1st details signal of the original signal associated with wavelet functions of: (a) Daub4; (b) Cheb4; (c) Geg4a12.

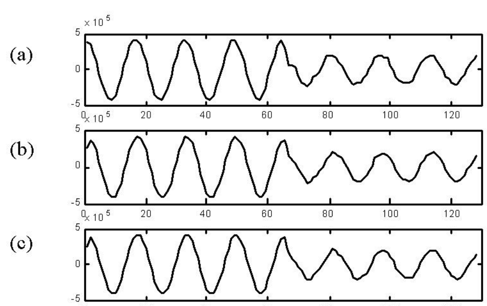


Fig. 10. 3rd approximating signal of the original signal associated with scaling functions of: (a) Daub4; (b) Cheb4; (c) Geg4a12.

6. CONCLUSIONS

A family of harmonic ultraspherical scaling and wavelet functions is introduced, which are related to Gegenbauer polynomials. Since α can assume several values ($\alpha \in \mathbb{R}$; $\alpha > 0$), there exist uncountable Gegenbauer scaling functions and mother wavelets, for any odd integer ν. These functions have compact support, which is very attractive from the practical viewpoint. This new family of wavelets encompasses the formalism derived in Haar, Legendre and Chebyshev wavelets. Additionally, the scaling filters have even symmetry and the wavelet filters have odd symmetry, and are, respectively, type II and type IV FIR generalized linear phase filters [9]. Each one has linear phase and so constant group delay, which means that there is no delay on different frequency components to compose the analysed signal. A further advantage of these wavelets is about their computational effort, since only half of its coefficients needs to be computed. The relevance of such new wavelets on fault analysis from power systems is currently being investigated.

7. REFERENCES

[1] R. J. de Sobral Cintra, H. M. de Oliveira, and L. R. Soares, “On filter banks and wavelets based on chebyshev polynomials,” in *Proc. of the 7th WSEAS Int. Conf. on Systems*, Corfu Island, Greece, July 2003.

[2] A. G. Deczky, “Unispherical windows,” in *Proc. of the IEEE Int. Symp. on Circuits and Systems (ISCAS2001)*, Sidney, Australia, May 2001, pp. 85–88.

[3] A. Rowinska-Schwarzweller and M. Wintermantel, “On designing fir filters using windows based on gegenbauer polynomials,” in *Proc. of the IEEE Int. Symp. on Circuits and Systems (ISCAS2002)*, Arizona, USA, May 2002, p. 413416.

[4] J. J. Soltis, “Analytic wavelets based upon de-orthogonalized gegenbauer polynomials,” in *Proc. of the IEEE-SP Int. Symp. on Time-Frequency and Time-Scale Analysis*, Philadelphia, PA, Oct. 1994, pp. 389–392.

[5] A. Saed, J. J. Soltis, and M. Ahmadi, “Gegenbauer (ultraspherical) polynomials for gabor-type wavelet approximation and fir filter function generation in wavelet analysis,” in *Proc. of the IEEE Canadian Conference on Electrical and Computer Engineering*, 1995, pp. 878–881.

[6] W. A. Wilkinson and M. D. Cox, “Discrete wavelet analysis of power system transients,” *IEEE Transactions on Power Delivery*, vol. 11, no. 4, pp. 2038–2044, 1996.

[7] I. Stegun M. Abramowitz, Ed., *Handbook of Mathematical Functions*, Dover, New York, 1968.

[8] C. K. Chui, *An Introduction to Wavelets*, Academic Press, San Diego, CA, 1992.

[9] A. V. Oppenheim, R. W. Schafer, and J. Buck, *Discrete-Time Signal Processing*, Prentice Hall, New Jersey, 1998.

[10] C. S. Burrus, R. A. Gopinath, and H. Guo, *Introduction to Wavelets and Wavelets Transforms: A Primer*, Prentice Hall, New Jersey, 1998.