THE ENTROPY OF A CODE WITH PROBABILITIES

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Abstract — The entropy of a code with probabilities is defined and as a consequence the concept of conservation of entropy in lossless coding emerges in a natural manner. For any given probability distribution \( (p_1, p_2, \ldots, p_T) \) all the distinct decompositions of the associated entropy function \( h(p_1, p_2, \ldots, p_T) \), as a function of entropies of lower order, are obtained from the terminal uncertainty of the distinct rooted trees with leaf probabilities \( (p_1, p_2, \ldots, p_T) \).

1. Introduction

In this paper we begin by reviewing some basic theory of rooted trees with probabilities and then define the entropy of a code with probabilities. As a consequence the concept of conservation of entropy in lossless coding emerges in a natural manner and we remark that, although implicitly a known result, the fact that entropy is preserved in lossless coding apparently has not been previously stated explicitly in the literature. We apply the concepts introduced to source coding as a matter of preference and not due to scope limitations.

2. Basics of Source Coding

Source coding is a well known technique used to represent the symbols at the output an information source with as few code digits per source symbol as possible. We will consider only lossless source coding in which case it is possible to reconstruct exactly the source output from its encoded representation. The source codes employed in lossless source coding are called uniquely decodable codes [1, p.48]. A sufficient condition for the unique decodability of a concatenation of codewords is that the encoding be prefix-free, i.e., that no codeword be the first part (prefix) of another codeword. This prefix-free condition is equivalent to the condition that a decoder be able to immediately recognize the end of a codeword without need to read the beginning of the next codeword. We will refer to codes with this property as prefix-free codes. For simplicity, we consider here only coding the output sequence \( U_1, U_2, U_3, \ldots, \) of an \( L \)-ary discrete memoryless source (DMS) [8, p.92], but the theory presented can be applied to sources with memory simply by replacing the probability distribution for \( U_i \) with the conditional probability distribution for \( U_i \) given the observed values \( U_1, U_2, \ldots, U_{i-1} \).

For the \( L \)-ary discrete memoryless source, the coding problem reduces to the problem of such coding for a single \( L \)-ary random variable \( U_i \). We concentrate our attention on binary coding both for its practical importance and because the generalizations to higher order alphabets are immediate.

3. Rooted Trees Revisited

Definition 1 A rooted tree with probabilities [8] is a finite rooted tree with probabilities assigned to each node such that

(a) The root node is assigned probability 1.
(b) The probability of every intermediate node (including the root node) is the sum of the probabilities of the nodes at depth 1 in the subtree stemming from this intermediate node.

An immediate result of Definition 1, known as the Path Length Lemma, was proved in [3] and goes as follows.

Proposition 1 In a rooted tree with probabilities, the average depth of the terminal nodes equals the sum of the probabilities of the intermediate nodes (including the root node).

Various uncertainties can naturally be defined in a rooted tree with probabilities. In [4] the probability of each node was defined as the probability that we would reach that node in a random journey through the tree starting at the root node and ending on some terminal node. Then, given that one is at a specified intermediate node, the conditional probability (the branching probability) of choosing each outgoing branch as the next leg of the journey is just the probability on the node at the end of that branch divided by the probability of the node at its beginning, i.e., the probability...
of this intermediate node itself. The branching uncertainty was defined [3, 4] at each intermediate node so that it equals the uncertainty of a random variable that specifies the branch followed out of that node, given that we had reached that node on our random journey.

**Definition 2** Suppose that \( q_{i1}, q_{i2}, \ldots, q_{iL} \) are the probabilities of the nodes (some of which may be intermediate nodes and some terminal nodes) at the ends of the \( L \) branches stemming outward from the intermediate node whose probability is \( P_i \). Then the branching uncertainty [3], \( H_i \), at this node is defined as

\[
H_i = - \sum_{j:q_{ij} \neq 0} \frac{q_{ij}}{P_i} \log \frac{q_{ij}}{P_i},
\]

because \( q_{ij}/P_i \) is the conditional probability of choosing the \( j \)th of these branches as the next leg on our journey given that we are at this node.

The average depth \( L \) of a rooted tree with \( T \) terminal nodes is given by the sum of \( T \) terms such that each term is the product of a terminal node probability by its depth from the root node.

After representing the output of a DMS by means of a prefix-free code we are often interested in computing the probability of single digits, or pairs of digits, etc., produced by the prefix-free code driven by the source. It turns out that the computation of such probabilities, directly from the associated rooted tree, is possible but becomes very complicated as longer blocks of code digits are considered, i.e., as the order of the statistics increases. We introduce next the code uncertainty and the code state diagram with probabilities. The latter provides a simpler way for computing code digit statistics.

Suppose that the code \( C \) has \( T \) codewords whose lengths are \( l_1, l_2, \ldots, l_T \) and whose associated probabilities are \( p_1, p_2, \ldots, p_T \).

### 4. Decomposition of Entropies

In what follows we use the definition of terminal uncertainty of a rooted tree with probabilities combined with Proposition 2 to establish all the distinct decompositions of an entropy function.

**Definition 3** The terminal uncertainty \( H_T [3] \) of the rooted tree is defined as the quantity

\[
H_T = - \sum_{i=1}^{T} p_i \log p_i.
\]

**Proposition 2** Let \( p_1, p_2, \ldots, p_T \) denote a probability distribution and let \( h(p_1, p_2, \ldots, p_T) \) denote the entropy in bits of this probability distribution. The number of distinct decompositions of \( h(p_1, p_2, \ldots, p_T) \) as a function of entropies involving \( p_1, p_2, \ldots, p_T \) is equal to the number of distinct rooted trees having \( T \) terminal nodes whose probabilities are \( p_1, p_2, \ldots, p_T \), respectively.

**Proof:** The proof follows directly from Theorem 1, [4], which states that

\[
H_T = \sum_{i=1}^{N} P_i H_i,
\]

where \( P_i \) is the node probability and \( H_i \) is the corresponding branching uncertainty.

### Example 1

Suppose that \( T = 4 \). Let \( h(x) \) denote the binary entropy, i.e., \( h(x) = -x \log x - (1 - x) \log(1 - x) \). There are essentially four distinct decompositions of \( h(p_1, p_2, p_3, p_4) \) as a function of entropies involving \( p_1, p_2, p_3 \) and \( p_4 \), namely

\[
\begin{align*}
\quad h(p_1, p_2, p_3, p_4) &= h(p_1, p_2, p_3, p_4), \\
&= h(p_1, p_2, p_3, p_4), \quad \text{(identity)} \\
\quad h(p_1, p_2, p_3, p_4) &= h(p_1) + \lambda_1 h(p_2, p_3, p_4, p_4/\lambda_1), \\
&= h(p_1) + \lambda_1 h(p_2, p_3, p_4) + \lambda_2 h(p_3, p_4), \\
\quad h(p_1, p_2, p_3, p_4) &= h(1 - \lambda_2) + \lambda_2 h(p_1, p_2, p_3, p_4, p_4/\lambda_2),
\end{align*}
\]

where \( \lambda_1 = 1 - p_1 \) and \( \lambda_2 = 1 - p_1 - p_2 \).

We would like to mention here that Proposition 2 is a generalization of Property 3 of the entropy function \( h(\cdot, \ldots) \) in Section 6 of [7] as exemplified above by the fourth decomposition of \( h(p_1, p_2, p_3, p_4) \).

### 5. The Entropy of a Code with Probabilities

After representing the output of a DMS by means of a prefix-free code we are often interested in computing the probability of single digits, pairs of
digits, etc., produced by the prefix-free code driven by the source. It turns out that the computation of such probabilities, directly from the associated rooted tree, is possible but becomes very complicated as longer blocks of code digits are considered, i.e., as the order of the statistics increases. We introduce next the code uncertainty and the code state diagram with probabilities. The latter provides a systematic way for computing code digit statistics.

**Definition 4** A code state diagram with probabilities is that state diagram obtained by modification of the code rooted tree as follows

(a) The tree terminal nodes are connected to the root node.

(b) The tree nodes are called states of the code state diagram with probabilities and the root node is called state 1.

(c) The probability $P_{e_i}$, $i = 1, 2, \ldots, N$, of every state $\sigma_i$ is equal to the probability of the corresponding tree node $P_i$ divided by the tree average depth $L$.

For prefix-free codes **Definition 4** implies that every codeword must start and end at state 1.

**Definition 5** The code uncertainty, $H_C$, of a code state diagram with probabilities is defined as the quantity

$$H_C = -\sum_{i=1}^{T} P_i \log P_i,$$

i.e., as the ratio between the associated rooted tree terminal uncertainty by the tree average depth.

We notice that our definition of code uncertainty is very general and is not limited to prefix-free codes. Actually, it is not even limited to unique decodability of the code considered.

**Example 2**

Suppose that $T = 4$ binary codewords have been numbered such that $l_1 = 1$, $l_2 = 2$, $l_3 = 3$ and $l_4 = 3$, and that $p_1 = .4$, $p_2 = .3$, $p_3 = .2$ and $p_4 = .1$, respectively. Then

$$H_C = -\sum_{i=1}^{T} P_i \log P_i = 1.5437 \quad 1.9 = .812.$$ 

**Theorem 1** The code uncertainty of a code state diagram with probabilities equals the sum over all states of the branching uncertainty at that state weighted by the state probability, i.e.,

$$H_C = \sum_{i=1}^{N} P_{e_i} H_i.$$ 

**Proof:** Each term $P_{e_i} H_i$ in the sum in (1) can be written as

$$P_{e_i} H_i = \frac{P_i}{L} \sum_{j \neq i} q_{ij} \log \frac{q_{ij}}{P_i}$$

$$= \frac{1}{L} \sum_{j \neq i} q_{ij} \log q_{ij} + \frac{1}{L} P_i \log P_i.$$ 

We notice that the term $\frac{1}{L} P_i \log P_i$ for every state $\sigma_i$, $i \neq 1$, is cancelled in the sum (1) by the term $-\frac{1}{L} P_i \log P_i$ originated by the state from which state $\sigma_i$ is reached. In this manner only terms of the form $-\frac{1}{L} \sum_{j \neq i} q_{ij} \log q_{ij}$, i.e., terms originating from those states connected to state $\sigma_i$ by a single branch, are not cancelled in the sum (1).

We notice that the $q_{ij}$'s, $1 \leq i \leq T$, are precisely the probabilities of the rooted tree terminal nodes. Hence we have proved that

$$H_C = \sum_{i=1}^{N} P_{e_i} H_i.$$

**Example 3**

Let us consider the code of Example 2. Figure 1 shows the corresponding code rooted tree and Figure 2 shows the associated code state diagram.

**6. Conservation of Entropy**

We now introduce the concept of Conservation of Entropy in Lossless Coding. Although implicitly a known result, the fact that entropy is preserved in lossless coding apparently has not been previously stated explicitly in the literature. Whenever a uniquely decodable code with an alphabet of $r$
symbols is used for the lossless coding of a DMS source $S$ with entropy $H(S)$, usually the ratio $\eta = H(S)/L \log r$ is the parameter of interest to measure the coding efficiency, where $L$ denotes the average codeword length. By our Definition 5 we observe that $H(S)/L$ is precisely the entropy of the source formed by the cascade of the original source and the encoder. The entropy of the encoded sequence plays an important role both in information theory and in cryptography.

**Proposition 3** Let $H(S)$ denote the entropy of a discrete memoryless source $S$. Consider that the output of $S$ is fed to an encoder $E_1$ which implements a lossless code $C_1$ with an $r_1$-ary alphabet and average codeword length $L_1$. The output of $E_1$ is then fed to the input of a second encoder $E_2$ which implements a lossless code $C_2$ with an $r_2$-ary alphabet and average codeword length $L_2$, etc. The following string of equalities hold true.

$$
H(S) = L_1 \log r_1 H_1(S) = L_2 \log r_2 H_2(S) = \ldots = L_k \log r_k H_k(S),
$$

where $H_i(S)$ and $L_i$, $1 \leq i \leq k$, denote respectively the entropy and the average codeword length for the $i$th code in the cascade.

The proof of Shannon's noiseless source coding theorem [8, p.60] implicitly suggests that the only way to reduce the redundancy of a message to be transmitted or stored is by extending the codewords of the original source. In [9] we introduced an iterative procedure to reduce redundancy, as an alternative to coding the extensions of a source. We remark that source coding is an efficient way to remove redundancy by performing data compression, in contrast to homophonic substitution [5, 6] which removes redundancy at the cost of data expansion.

**References**


