Non-ergodicity and growth are compatible for 1-D local interaction

A. D. Ramos\(^1\) A. Toom\(^1\)

Abstract

We present results of Monte Carlo simulation and chaos approximation of a class of Markov chains with a countable or continuous set of states. Each of these states can be written as a finite or bi-infinite sequence of *pluses* and *minuses* denoted by \(\oplus\) and \(\ominus\). As continuous time goes on, our sequence undergoes the following three types of local transformation: The first one, called *flip*, changes any minus into plus and any plus into minus with a rate \(\beta\). Another, called *annihilation*, eliminates two neighbor components with a rate \(\alpha\) whenever they are in different states. The third one, called *mitosis*, doubles any component with a rate \(\gamma\). All of them occur at any place of the sequence independently. Our simulation and approximation have shown that with appropriate positive \(\alpha\), \(\beta\) and \(\gamma\) this process has the following two properties. **Growth:** In the finite length case, as the process goes on, the length of the sequence tends to infinity with a probability, which tends to 1 when the length of the initial sequence tends to \(\infty\). **Non-ergodicity:** The process with infinite length is non-ergodic and the process with finite length shows some analog of non-ergodicity.

**Key words:** local interaction; cellular automata; particle process; Monte Carlo simulation; chaos approximation; phase transitions; positive rates conjecture; variable length.

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Introduction

Since the first studies of Ising model, it became common among physicists to recognize the qualitative difference between one-dimensional and multi-dimensional case for all multi-component models with local interaction. This lore crystallized in the shape of the “positive rates conjecture” (see [4, pp. 178, 201]) and was brilliantly refuted by Peter Gács ([2, 3]). However, cases, when a random process with one-dimensional local interaction shows some form of non-ergodicity, remain non-trivial and for this reason still attract attention; our task is to provide another case of this sort.

We consider two cases at once: finite and infinite. In the infinite case configurations are bi-infinite sequences of pluses and minuses denoted by \( \oplus \) and \( \ominus \) respectively. In the finite case, to avoid complications at the ends, we use configurations called “circulars”. A circular is just a finite sequence of pluses and minuses, but terms of this sequence, called components, are enumerated by remainders modulo \(|C|\), where \(|C|\) is the length (that is the number of components) of the circular, rather than natural numbers. (In the literature this is called sometimes periodic condition.) Figure 1 shows a circular \( C \) with length \( n = |C| \) and components \( C_0, \ldots, C_{n-1} \), whose indices \( 0, \ldots, n-1 \) are remainders modulo \( n \).

![Figure 1: A circular \( C \) with \( |C| = n \).](image)

The usual finite sequences of pluses and minuses, whose terms are indexed by natural numbers, are called words. The length of a word \( W \) is denoted by \(|W|\). There is the empty word, denoted by \( \Lambda \), whose length is zero. We say that a word \( W = (a_1, a_2, \ldots, a_n) \) appears at a place \( i \) in a circular \( C = (C_0, \ldots, C_{n-1}) \)
if $C_{i+1} = a_1$, $C_{i+2} = a_2$, ..., $C_{i+n} = a_n$. If a word $W$ appears in a circular $C$ and $|W| \leq |C|$, we can substitute it by another word $V$, thus obtaining another circular. We shall label such a substitution with $W \rightarrow V$. Our process consists in application of three concrete kinds of substitutions. Namely, as continuous time goes on, our sequence (finite or infinite) undergoes the following types of transformation:

- **Annihilation**: $(\oplus, \ominus) \rightarrow \Lambda$ and $(\ominus, \oplus) \rightarrow \Lambda$. If the states of the components with indices $x$ and $x+1$ are different, both disappear with a rate $\alpha$ independently of the other components. The components $x-1$ and $x+2$ become neighbours. The length of the circular decreases by two.

- **Flip**: $\oplus \rightarrow \ominus$ and $\ominus \rightarrow \oplus$. This changes the state of one component with a rate $\beta$ independently of the other components. The length of the circular does not change.

- **Mitosis**: $\oplus \rightarrow \oplus\oplus$ and $\ominus \rightarrow \ominus\ominus$. This duplicates one component with a rate $\gamma$ independently of other components. The length of the circular increases by one.

In the finite case the text presented above may be accepted as a definition. In the infinite case the corresponding class of processes has never been rigorously defined (according to our knowledge). However, we have reasons to believe that such a definition is possible. Processes of this sort have been mentioned in a physical context [5, 6]. For similar processes with discrete time, some special cases have been already studied [7, 8, 9, 11] and a general definition will be available soon [10]. Thus we take the liberty to speak about both finite-space and infinite-space versions of the processes described above and study them using Monte Carlo method and chaos approximation.
Our main result: both Monte Carlo simulation and chaos approximation have shown that with appropriate positive $\alpha$, $\beta$ and $\gamma$ this process has two properties:

**Growth:** In the finite case the length of the sequence tends to infinity with a probability, which tends to 1 when the length of the initial circular tends to $\infty$.

**Non-ergodicity:** the infinite process is non-ergodic and the finite process shows some analog of non-ergodicity.

Our work was motivated by success and failure of [11], which considered infinite processes similar to ours with these differences: time was discrete (which we deem unimportant), flip was asymmetric, that is it turned minuses into pluses, but not vice versa (which also is unimportant for us) and mitosis was absent (which is important). [11] proved some form of non-ergodicity for that process for $\alpha$ small enough: if the process started with “all minuses”, the percentage of pluses always remained small. This was a success and it was improved in [8] and studies numerically in [9]. The failure of [11] was impossibility to present a finite analog: in the absence of mitosis, the length of the sequence decreased in average and the configuration degenerated. In our work this failure is removed.

**Monte Carlo simulation**

By its very nature, the Monte Carlo method always refers to some finite space case, even when the ultimate motivation is to study the infinite case. In addition, even when we study a continuous time process, its computer simulation always has discrete time. This applies to our study also. Thus, our approximation is a Markov chain with a countable set $\Omega$ of states, where $\Omega$ is the set of circulars of all lengths (in fact, within a certain finite range of lengths). The time $t$ (that is, the number of iterations of our computer simulation) is discrete and at every time step at most one transformation of the list (1), chosen at random, takes place. Thus, in each individual experiment we obtain a randomly generated sequence of circulars
and the circular obtained at time $t$ is denoted by $C^t$. Its $x$-th component is denoted by $C^t_x$, where $x = 0, \ldots, |C^t| - 1$. We denote by $\mathcal{M}$ the set of probability distribution, that is the set of normalized measures on $\Omega$, the set of circulars. We consider the circulars $C^t$ as representations of measures $\mu^t \in \mathcal{M}$, so the sequence $C^0, C^1, C^2, \ldots$, is a trajectory of some random process $\mu^0, \mu^1, \mu^2, \ldots$.

Let us denote by $\text{quant}(W | C)$ the quantity of different places, at which a word $W$ appears in a circular $C$. After that we define the frequency of $W$ in $C$ as

$$\text{freq}(W | C) = \frac{\text{quant}(W | C)}{|C|}. \quad (2)$$

For any $\mu \in \mathcal{M}$ we define the frequency of the word $W$ according to $\mu$ as

$$\text{freq}(W | \mu) = \sum_{C \in \Omega} \text{freq}(W | C) \cdot \mu(C). \quad (3)$$

We were especially interested in $\text{freq}(\oplus | \mu^t)$, that is the frequency of pluses at time $t$. Due to limitations of our computer facilities, we could not estimate $\text{freq}(\oplus | \mu^t)$ directly, so we approximated it by

$$\overline{\text{freq}(\oplus | \mu^t)} \overset{\text{def}}{=} \frac{1}{t} \sum_{k=1}^{t} \text{freq}(\oplus | C^k). \quad (4)$$

Since the rules of our process do not change when we swap plus and minus, the ergodicity of our infinite process implies that the frequency of pluses tends to $1/2$ and for its finite analog this frequency tends to $1/2$ with a probability, which tends to 1 when the length of the initial condition tends to $\infty$. Of course, all finite systems are ergodic, but we wanted to make conclusions about infinite space processes. So let us describe a procedure, which we call *Imitation*. This pro-
procedure generates a sequence of circulars in the following inductive way. (Forget imprefections of computer generated random variables.)

**Base of induction.** The initial circular $C^0$ consists of 1000 minuses.

**$t$-th induction step.** Given a circular $C^t$, where $t = 0, 1, 2, \ldots$ we performed these three procedures:

*The first procedure imitated the random choice of a place where to perform a transformation:* a random integer $x$ distributed uniformly in $\{0, 1, \ldots, |C^t| - 1\}$ was generated to identify the position, where the transformation would occur.

*The second procedure imitated (1):* it generated a real random $\xi$ distributed uniformly in $(0, 1)$. Then:

- if $\xi \in \left[0, \frac{\alpha}{\alpha + \beta + \gamma}\right)$ and $C^t_x \neq C^t_{x+1}$, these components annihilated, that is both of them disappeared.

- If $\xi \in \left[\frac{\alpha}{\alpha + \beta + \gamma}, \frac{\alpha + \beta}{\alpha + \beta + \gamma}\right)$, the component $C^t_x$ changed its state from $\ominus$ to $\oplus$ or from $\oplus$ to $\ominus$.

- If $\xi \in \left[\frac{\alpha + \beta}{\alpha + \beta + \gamma}, 1\right)$, this component underwent mitosis, that is turned into two components in the same state.

Let us denote the resulting circular by $(C')^t$. Due to presence of annihilation and mitosis, the length of $(C')^t$ might be different from the length of $C^t$; so in the course of the process the length of our circular changed randomly and usually had a tendency either to shrink all the time or to grow all the time. To prevent our process from shrinking to degeneration or growing beyond our computer possibilities we used the third procedure.
The third procedure which helped to keep $C^t$ within range: given $(C')^t$, we generated a new circular, namely $C^{t+1}$, in one of the following ways. **Double:** if $|(C')^t| < N_{\text{min}}$, where $N_{\text{min}} = 500$, then $C^{t+1}$ was obtained from $(C')^t$ by concatenating it with its copy and thereby duplicating its length. **Cut:** if $|(C')^t| > N_{\text{max}}$, where $N_{\text{max}} = 15,000$, then $C^{t+1}$ was obtained from $(C')^t$ deleting half of it. Otherwise we changed nothing and obtained $C^{t+1} = (C')^t$.

**When we stop:** We stopped our simulation when each one of the three transformations (1) occurred at least 100,000 times. Thus the procedure Imitation is described.

To obtain the small squares on Figure 2, approximating the boundary between the regions of ergodicity and non-ergodicity, we used Imitation to attribute an appropriate value to a Boolean variable denoted by $E$ (which means ergodicity) as follows: if at the end of iteration the quantity $\overline{\text{freq}}(\oplus|\mu^t)$ was in the range $(0.45, 0.55)$, we set $E = yes$; otherwise we set $E = no$. We interpret the result $E = yes$ as a suggestion that the infinite process with the triple $(\alpha, \beta, \gamma)$ is ergodic; the result $E = no$ was interpreted as a suggestion that this triple produces a non-ergodic process. We used Imitation within a cycle with growing $\alpha/\beta$: we started with $\alpha/\beta = 0.1$ and then iteratively performed Imitation and increased $\alpha/\beta$ by 0.1 and repeated this until $\alpha/\beta$ reached the value 8 or $E$ got the value $no$. Thus we obtained a certain value of $\alpha/\beta$. In fact, we performed this cycle 5 times and recorded the arithmetical average of the 5 values of $\alpha/\beta$ thus obtained. All this was done for 50 values of $\gamma/\beta$, namely the values $\gamma_i/\beta = 0.1 \times i$ with $i = 1, \ldots, 50$. Thus we obtained 50 pairs $(\alpha_i/\beta, \gamma_i/\beta)$ represented by small squares on Figure 2.

To obtain the small circles on Figure 2, approximating the boundary between the regions of growth and shrinking, we used Imitation within a cycle with growing
\(\gamma/\beta\): we started with \(\gamma/\beta = 0.1\) and then iteratively performed Imitation and increased \(\gamma/\beta\) by 0.1 and repeated this until \(\gamma/\beta\) reached the value 8 or there was none duplication in the course of performing Imitation. Thus we obtained a certain value of \(\gamma/\beta\). In fact, we performed this cycle 5 times and recorded the arithmetical average of the 5 values of \(\gamma/\beta\) thus obtained. All this was done with 17 values of \(\alpha/\beta\), namely the values \(\alpha_i/\beta = 0.5 \times i\) with \(i = 0, \ldots, 16\). Thus we obtained 17 pairs \((\alpha_i/\beta, \gamma_i/\beta)\) represented by small circles on Figure 2.

![Figure 2](image-url)

**Figure 2:** White squares approximate the boundary between suggested ergodicity and suggested non-ergodicity. White balls approximate the boundary between shrinking and growing. Compare this figure with figure 7.

Now let us look at Figure 3. Our simulations suggest that \(\text{freq}(\oplus|C^t)\) may behave in two qualitatively different ways: For some \((\alpha, \beta, \gamma)\) the frequency of pluses kept close to 1/2 as shown by those points on 3a), whose vertical coordinates are between 0.3 and 0.7. However, for some other triples \((\alpha, \beta, \gamma)\) the frequency of pluses oscillated all the time of simulation as shown on Figure 3b). More specifically, most of the time the frequency of pluses was either near a number close to zero or near a number close to one. We suggest that when \(\text{freq}(\oplus|C^t)\) oscillates, the corresponding infinite process is non-ergodic. The points on Figure 3a), whose vertical coordinates are less than 0.2, pertain to the case \(\alpha = 70, \beta = 10\) and
$\gamma = 5$. Although oscillation seems to be absent in this case, we suggest that it is present here also, but the average time of keeping at one extreme is too large to be visible within our computer time. Thus we suggest to classify this case together with those on Figure 3b).
Figure 3: a) The points, whose vertical coordinates are between 0.3 and 0.7 pertain to the case \( \alpha = \beta = 10 \) and \( \gamma = 30 \). We believe that in this case our infinite process is ergodic. The points with the vertical coordinates less than 0.2 pertain to the case \( \alpha = 70, \beta = 10 \) and \( \gamma = 5 \). We conjecture that in this case our infinite process is non-ergodic. The figure b) pertains to the case \( \alpha = 70, \beta = 10 \) and \( \gamma = 30 \). We observe that \( \text{freq}(\oplus | C^t) \) spends most of the time near 0.1 or 0.9, sometimes rapidly swinging from one extreme to the other. We suggest that the corresponding infinite system is non-ergodic.
Chaos approximation

Now let us consider a deterministic approximation of our process, which we call chaos approximation. Let $X(t)$ and $Y(t)$ denote some real quantities approximating the quantities of $\oplus$ and $\ominus$ respectively at time $t$. The following differential equations approximate our random process assuming that the particles are mixed all the time:

$$
\begin{align*}
\frac{dX(t)}{dt} &= -\beta \cdot X(t) + \beta \cdot Y(t) + \gamma \cdot X(t) - \alpha \cdot \frac{X(t)Y(t)}{X(t) + Y(t)}, \\
\frac{dY(t)}{dt} &= -\beta \cdot Y(t) + \beta \cdot X(t) + \gamma \cdot Y(t) - \alpha \cdot \frac{X(t)Y(t)}{X(t) + Y(t)}.
\end{align*}
$$

We may go to other variables

$$
S(t) = X(t) + Y(t) \quad \text{and} \quad B(t) = \frac{X(t) - Y(t)}{X(t) + Y(t)}.
$$

For simplicity, sometimes we shall denote $X(t), Y(t), S(t)$ and $B(t)$ by $X, Y, S$ and $B$ respectively. The following system of equations is equivalent to (5):

$$
\begin{align*}
\frac{dS}{dt} &= S \cdot \left( \gamma - \frac{\alpha}{2} \left( 1 - B^2 \right) \right), \\
\frac{dB}{dt} &= B \cdot \left( \frac{\alpha}{2} \left( 1 - B^2 \right) - 2\beta \right).
\end{align*}
$$

This system is easy to solve explicitly, but we shall get all we need by qualitative arguments. Since we are especially interested in the proportion of each type of particles, we consider also another process, which we call normalized chaos approximation:

$$
X_{\text{norm}}(t) = \frac{X(t)}{X(t) + Y(t)}, \quad Y_{\text{norm}}(t) = \frac{Y(t)}{X(t) + Y(t)}.
$$
Since the original process (5) is homogeneous, any piece of it commutes with normalization. So we deal with a two-dimensional analog of the theorem on p. 7 of [1]. Thus, to deduce an equation for (9), we may break the process (5) into small steps and after every step we may normalize the resulting point, that is project it to the line $X + Y \equiv 1$ as shown in figure 4. Then we go to the limit as the steps become smaller and their number grows.

**Figure 4.** An explanation of how the chaos process turns into the normalized chaos process. We break the chaos process into small steps and after every step the resulting point is normalized, that is projected to the line $X + Y \equiv 1$.

In the normalized case we may restrict our attention to only one independent variable; as such we choose $B(t)$ - the same as we defined in (6) and come to the equation (8) again. Since the value of $B$ does not change under projection,
\[ B(t) = \frac{X(t) - Y(t)}{X(t) + Y(t)} = \frac{X_{\text{norm}}(t) - Y_{\text{norm}}(t)}{X_{\text{norm}}(t) + Y_{\text{norm}}(t)} = X_{\text{norm}}(t) - Y_{\text{norm}}(t), \]

which is equivalent to

\[ X_{\text{norm}}(t) = \frac{1 + B(t)}{2} \quad \text{and} \quad Y_{\text{norm}}(t) = \frac{1 - B(t)}{2}. \]

So we may treat (8) as a deterministic dynamical system with a space \([-1, 1]\) and continuous time \(t\). We call a number \(B^* \in [-1, 1]\) a fixed point of this system if (8) equals zero at \(B = B^*\). We say that a fixed point \(B^* \in [-1, 1]\) attracts a point \(B \in [-1, 1]\) if the process (8) starting at \(B(0) = B\) tends to \(B^*\) when \(t \to \infty\). Given a fixed point, we call its basin of attraction or just basin the set of points attracted by it. Let us study fixed points and their basins for (8). The right side of (8) equals zero at three (generally complex) values of \(B\), which we denote by

\[ B_1^* = -\sqrt{1 - \frac{4\beta}{\alpha}}, \quad B_2^* = 0, \quad B_3^* = \sqrt{1 - \frac{4\beta}{\alpha}}. \quad (10) \]

Hence follows this classification:

If \(\alpha < 4\beta\), then \(B_1^*\) and \(B_3^*\) are not real and the right side of (8) is

\[
\begin{align*}
\begin{cases}
\text{positive when} & B \in [-1, 0), \\
\text{zero when} & B = 0, \\
\text{negative when} & B \in (0, 1].
\end{cases}
\end{align*}
\]

The following scheme illustrates this:
Figure 5. Behavior of $B(t)$ when $\alpha < 4\beta$.

Therefore in this case $B(t)$ tends to zero from any initial value when $t \to \infty$.

If $\alpha = 4\beta$, then $B^*_1$ and $B^*_3$ are real and equal to zero. The signs of the right side of (8) are the same as in the previous case and $B(t)$ also tends to zero from any initial condition when $t \to \infty$.

If $\alpha > 4\beta$, then $B^*_1$ and $B^*_3$ are real and $-1 < B^*_1 < B^*_2 = 0 < B^*_3 < 1$ (remember that $\beta > 0$). So the right side of (8) is

$$
\begin{cases}
\text{positive when } B \in [-1, B^*_1), \\
\text{zero when } B = B^*_1, \\
\text{negative when } B \in (B^*_1, B^*_2), \\
\text{zero when } B = B^*_2 = 0, \\
\text{positive when } B \in (B^*_2, B^*_3), \\
\text{zero when } B = B^*_3, \\
\text{negative when } B \in (B^*_3, 1].
\end{cases}
$$

The following scheme illustrates this:

![Figure 6. Behavior of $B(t)$ when $\alpha > 4\beta$.](image)

Therefore in this case $B(t)$ tends to $B^*_1$ or $B^*_2$ or $B^*_3$ from any initial condition when $t \to \infty$, so $[-1, 1]$ is a union of three basins:

$$
\text{basin}(B^*_1) = [-1, 0), \quad \text{basin}(B^*_2) = \{0\}, \quad \text{basin}(B^*_3) = (0, 1].
$$

Thus the normalized chaos approximation is ergodic if $\alpha \leq 4\beta$ and non-ergodic if $\alpha > 4\beta$. 
Now we are ready to study the chaos approximation (5). Let us remember that 
\( X(t) + Y(t) = S(t) \) and say that our dynamical system:

- **grows** if \( S(t) \) tends to infinity when \( t \to \infty \).
- **shrinks** if \( S(t) \) tends to zero when \( t \to \infty \).

Let us find out when it grows and when it shrinks.

Notice that we may rewrite (7) as

\[
\frac{d \ln S}{dt} = \gamma - \frac{\alpha}{2} (1 - B^2). \tag{11}
\]

Let us denote by \( G(B) \) the right side of (11).

Given two functions \( f_1 \) and \( f_2 \) of \( t \geq 0 \), let us write \( f_1 \sim f_2 \) if there are positive numbers \( t_0 \) and \( C \) such that

\[
\forall t \geq t_0 : 0 < f_1(t) < C \cdot f_2(t) \quad \text{and} \quad 0 < f_2(t) < C \cdot f_1(t).
\]

**Lemma.** Let \( B(0) \in \text{Basin}(B_i^*) \), where \( i \in \{1, 2, 3\} \). Then:

If \( G(B_i^*) > 0 \), then \( \ln S(t) \sim t \).

If \( G(B_i^*) = 0 \), then \( |\ln S(t)| = o(t) \).

If \( G(B_i^*) < 0 \), then \( -\ln S(t) \sim t \).

**Proof.** The easiest case is when the initial value of \( B \) is a fixed point of (8). In this case \( B(t) \equiv B(0) \) for all \( t \geq 0 \), so the right side of (11) is a constant.

If \( B(0) \) equals \( B_2^* = 0 \), then (11) turns into

\[
\frac{d \ln S(t)}{dt} = \gamma - \alpha/2,
\]

whence \( \ln S(t) = (\gamma - \alpha/2) \cdot t + \text{const} \), so

\[
S(t) = e^{(\gamma - \alpha/2) \cdot t} \times \text{a constant}.
\]
Therefore our process grows if $\gamma > \alpha/2$ and shrinks if $\gamma < \alpha/2$.

If $B_1^*$ and $B_3^*$ are real and $B(0)$ equals one of them, then (11) turns into

$$\frac{d \ln S(t)}{dt} = \gamma - 2\beta,$$

whence $\ln S(t) = (\gamma - 2\beta) \cdot t + \text{const},$ so

$$S(t) = e^{(\gamma - 2\beta) t} \text{ times a constant.}$$

Therefore our process grows if $\gamma > 2\beta$ and shrinks if $\gamma < 2\beta$.

Now let us consider the general case: $B(0)$ is any number in $[-1, 1]$. Then $B(0)$ belongs to a basin of some $B_i^*$, where $i \in \{1, 2, 3\}$. Then from (11) for any $t \geq 0$

$$\ln S(T) = \ln S(0) + \int_0^T G(B(t)) \, dt,$$  \hfill (12)

where the integration is taken along the trajectory $(S(t), B(t))$ of our process. Let us prove that

$$\frac{\ln S(t)}{t} \text{ tends to } G(B(b_i)) \text{ when } t \to \infty. \hfill (13)$$

This fact is similar to the well-know fact that if a sequence has a limit, its Cesàro transformation has the same limit. So the idea of the proof is the same. Notice that the function $G$ is limited and continuous on $[-1, 1]$. Let us take a function $f(T)$, which is small by comparison with $T$, but large by comparison with 1 (for example, it may be $f(T) = \sqrt{T}$) and divide (12) by $t$:

$$\frac{1}{t} \cdot \ln S(t) = \frac{1}{t} \cdot \ln S(0) + \frac{1}{t} \cdot \int_0^{f(T)} G(B(t)) \, dt + \frac{1}{t} \cdot \int_{f(T)}^T G(B(t)) \, dt.$$

Let $t \to \infty$. Then in the right part of this equation the first term evidently tends to zero, the second term tends to zero since the function $G$ is limited and the last term tends to $G(B_i^*)$ since the function $G$ is continuos. This proves (13) and our lemma. The following diagrams resume our findings.
Figure 7. Classification for $X(0) \neq Y(0)$. Compare this figure with figure 2.

In the special case when $X(0) = Y(0)$, $B(t) = 0$ for all $t$. But, zero is a fixed point, so the process is ergodic.

**Conclusion.**

Our purpose was to show that our class of processes with three positive parameters $\alpha, \beta, \gamma$ can grow and be non-ergodic at the same time. Each of our methods shows this in some area of our parameter space. So we have another example of 1-D non-ergodicity.

**Acknowledgments.** A. D. Ramos and A. Toom cordially thank CNPq, respectively MCT/CT-Info/550844/2007-4 and 301266/2007-7, which supported this work.
References


